Math 290 Name: Key Dr. Beezer
Exam 6 Spring 2016
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. Unless the problem directions specify otherwise, you may use Sage to manipulate matrices and vectors, row-reduce matrices, compute determinants, compute and factor characteristic polynomials, and compute eigenvalues and eigenspaces. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features of something I can't see).

1. Consider the linear transformation T below, between P_1, the space of polynomials of degree at most 1 and M_{12}, the space of 1×2 matrices. Prove that T is a linear transformation. (15 points)

$$T\colon P_1 \to M_{12}, \qquad T(a+bx) = \begin{bmatrix} a+2b & 3a-b \end{bmatrix}$$

① $T((a+bx)+(c+dx)) = T((a+c)+(b+d)x) = [(a+c)+2(b+d) \;\; 3(a+c)-(b+d)]$
$= [(a+2b)+(c+2d) \;\; (3a-b)+(3c-d)] = [a+2b \;\; 3a-b] + [c+2d \;\; 3c-d]$
$= T(a+bx) + T(c+dx)$

② $T(\alpha(a+bx)) = T(\alpha a + (\alpha b)x) = [\alpha a + 2(\alpha b) \;\; 3(\alpha a) - \alpha b]$
$= [\alpha(a+2b) \;\; \alpha(3a-b)] = \alpha[a+2b \;\; 3a-b] = \alpha T(a+bx)$

2. Consider the linear transformation S below, between P_2, the space of polynomials of degree at most 2, and $\mathbb{C}^3$, the vector space of column vectors with three entries. Compute the kernel and range, $\mathcal{K}(S)$ and $\mathcal{R}(S)$. (15 points)

$$S\colon P_2 \to \mathbb{C}^3, \qquad S(a+bx+cx^2) = \begin{bmatrix} a+b-c \\ 3a+4b-5c \\ 3a+3c \end{bmatrix}$$

$\mathcal{K}(S)$

$S(a+bx+cx^2) = \mathbf{0}$

$$\begin{bmatrix} a+b-c \\ 3a+4b-5c \\ 3a+3c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Homogenous system, solutions from rref

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} a = -c \\ b = 2c \end{matrix}$$

$(-c) + (2c)x + cx^2$
$= c(-1+2x+x^2)$

$\mathcal{R}(S)$ Spanned by S on a basis
$\{S(1), S(x), S(x^2)\} = \left\{\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 3 \end{bmatrix}\right\}$
This is a linearly dependent set we can improve (we expect rank of S to be 2!)
BCS says. 3rd column vector is surplus.

Answer:
$\mathcal{K}(S) = \langle\{-1+2x+x^2\}\rangle$; $\mathcal{R}(S) = \left\langle\left\{\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}\right\}\right\rangle$

3. Consider the linear transformation T below, between P_2, the space of polynomials of degree at most 2, and $\mathbb{C}^3$, the vector space of column vectors with three entries. Prove that T is invertible without using your work in subsequent questions as justification. (15 points)

$$T\colon P_2 \to \mathbb{C}^3, \qquad T\left(a + bx + cx^2\right) = \begin{bmatrix} 3a + 2b + 7c \\ 2a + b + 7c \\ -a - b - c \end{bmatrix}$$

Several approaches, simplest is $n(T) = 0$. So $K(T) = ?$

$T(a + bx + cx^2) = \underline{0}$

$\Rightarrow \begin{bmatrix} 3a+2b+7c \\ 2a+b+7c \\ -a-b-c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ Homogenous system, only solution $a = b = c = 0$.

$K(T) = \{0 + 0x + 0x^2\} \Rightarrow n(T) = 0 \Rightarrow$ injective

$r(T) + n(T) = \dim(P_2) \Rightarrow r(T) + 0 = 3 \Rightarrow r(T) = 3$

$\Rightarrow R(T) = \mathbb{C}^3 \Rightarrow T$ surjective

By Theorem ILTIS, T is invertible.

4. Using the same T as at the top of this page, find an explicit formula for T^{-1}. (25 points)

Pre images of basis vectors for codomain, $\mathbb{C}^3$.

$T^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \{6 - 5x - x^2\}$

$T^{-1}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \{-5 - 4x - x^2\}$

$T^{-1}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \{7 - x - x^2\}$

$$T^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = T^{-1}\left(a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= a\,T^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + b\,T^{-1}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + c\,T^{-1}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= a(6 - 5x - x^2) + b(-5 - 4x - x^2) + c(7 - x - x^2)$$

$$= (6a - 5b + 7c) + (-5a - 4b - c)x + (-a - b - c)x^2$$

Answer:

=

5. Suppose that A is an $m \times n$ matrix and define the function $T\colon \mathbb{C}^m \to \mathbb{C}^n$ by $T(\mathbf{x}) = A\mathbf{x}$. Verify that T meets the *definition* of a linear transformation. (10 points)

1) $T(\underset{\sim}{x}_1 + \underset{\sim}{x}_2) = A(\underset{\sim}{x}_1 + \underset{\sim}{x}_2)$

$= A\underset{\sim}{x}_1 + A\underset{\sim}{x}_2$ MMDAA

$= T(\underset{\sim}{x}_1) + T(\underset{\sim}{x}_2)$

2) $T(\alpha \underset{\sim}{x}) = A(\alpha \underset{\sim}{x})$

$= \alpha(A\underline{x})$ MMSMM

$= \alpha\, T(\underset{\sim}{x})$

6. Adjust the previous problem so that A is now an invertible $n \times n$ matrix. As before, define the function $T\colon \mathbb{C}^n \to \mathbb{C}^n$ by $T(\mathbf{x}) = A\mathbf{x}$. Verify that T meets the *definition* of an invertible linear transformation. (5 points)

Define $S: \mathbb{C}^n \to \mathbb{C}^n$ by $S(\underset{\sim}{x}) = A^{-1}\underset{\sim}{x}$ exists!

① S <u>is</u> a linear transformation by Problem 6.

② $(S \circ T)(\underset{\sim}{x}) = S(T(\underset{\sim}{x})) = S(A\underset{\sim}{x}) = A^{-1}(A\underset{\sim}{x}) = (A^{-1}A)\underset{\sim}{x} = I\underset{\sim}{x} = \underset{\sim}{x}$

③ $(T \circ S)(\underset{\sim}{x}) = T(S(\underset{\sim}{x})) = T(A^{-1}\underset{\sim}{x}) = A(A^{-1}\underset{\sim}{x}) = (AA^{-1})\underset{\sim}{x} = I\underset{\sim}{x} = \underset{\sim}{x}$

Since both compositions are the identity function, $S = T^{-1}$.

7. Suppose that $S\colon U \to V$ is a surjective linear transformation and that $R = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}$ spans the vector space U. Prove that the set $T = \{S(\mathbf{u}_1), S(\mathbf{u}_2), S(\mathbf{u}_3), \ldots, S(\mathbf{u}_k)\}$ spans the vector space V. (15 points)

① Grab an <u>output</u> $\underset{\sim}{v} \in V$.

② S surjective $\Rightarrow$ input $\underset{\sim}{u} \in U$ so that $S(\underset{\sim}{u}) = \underset{\sim}{v}$.

③ R spans U $\Rightarrow$ scalars a_i so that $\underset{\sim}{u} = a_1\underset{\sim}{u}_1 + a_2\underset{\sim}{u}_2 + \cdots + a_k\underset{\sim}{u}_k$.

④ $\underset{\sim}{v} \overset{②}{=} S(\underset{\sim}{u}) \overset{③}{=} S(a_1\underset{\sim}{u}_1 + a_2\underset{\sim}{u}_2 + \cdots + a_k\underset{\sim}{u}_k)$

$= a_1 S(\underset{\sim}{u}_1) + a_2 S(\underset{\sim}{u}_2) + \cdots + a_k S(\underset{\sim}{u}_k)$ LTLC

This result says T spans V.