

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.

For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. Unless the problem directions specify otherwise, you may use Sage to manipulate matrices and vectors, row-reduce matrices, compute determinants, compute and factor characteristic polynomials, and compute eigenvalues and eigenspaces. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features of something I can't see).

1. Consider the linear transformation  $T$  below, between  $P_1$ , the space of polynomials of degree at most 1 and  $M_{12}$ , the space of  $1 \times 2$  matrices. Prove that  $T$  is a linear transformation. (15 points)

$$T: P_1 \rightarrow M_{12}, \quad T(a + bx) = [a + 2b \quad 3a - b]$$

$$\begin{aligned} \textcircled{1} \quad T((a+bx) + (c+dx)) &= T((a+c) + (b+d)x) = [(a+c) + 2(b+d) \quad 3(a+c) - (b+d)] \\ &= [(a+2b) + (c+2d) \quad (3a-b) + (3c-d)] = [a+2b \quad 3a-b] + [c+2d \quad 3c-d] \\ &= T(a+bx) + T(c+dx) \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad T(\alpha(a+bx)) &= T(\alpha a + (\alpha b)x) = [\alpha a + 2(\alpha b) \quad 3(\alpha a) - \alpha b] \\ &= [\alpha(a+2b) \quad \alpha(3a-b)] = \alpha[a+2b \quad 3a-b] = \alpha T(a+bx) \end{aligned}$$

2. Consider the linear transformation  $S$  below, between  $P_2$ , the space of polynomials of degree at most 2, and  $\mathbb{C}^3$ , the vector space of column vectors with three entries. Compute the kernel and range,  $\mathcal{K}(S)$  and  $\mathcal{R}(S)$ . (15 points)

$$S: P_2 \rightarrow \mathbb{C}^3, \quad S(a + bx + cx^2) = \begin{bmatrix} a + b - c \\ 3a + 4b - 5c \\ 3a + 3c \end{bmatrix}$$

$\mathcal{K}(S)$

$$\begin{aligned} S(a+bx+cx^2) &= \underline{0} \\ \begin{bmatrix} a+b-c \\ 3a+4b-5c \\ 3a+3c \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Homogeneous system, solutions from rref

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} a &= -c \\ b &= 2c \end{aligned}$$

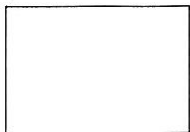
$$\begin{aligned} (-c) + (2c)x + cx^2 \\ = c(-1 + 2x + x^2) \end{aligned}$$

$\mathcal{R}(S)$

spanned by  $S$  on a basis  
 $\{S(1), S(x), S(x^2)\} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 3 \end{bmatrix} \right\}$   
 This is a linearly dependent set we can improve (we expect rank of  $S$  to be 2!)  
 BCS says. 3<sup>rd</sup> column vector is surplus.

Answer:

$$\mathcal{K}(S) = \langle -1 + 2x + x^2 \rangle; \quad \mathcal{R}(S) = \left\langle \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \right\rangle$$



3. Consider the linear transformation  $T$  below, between  $P_2$ , the space of polynomials of degree at most 2, and  $\mathbb{C}^3$ , the vector space of column vectors with three entries. Prove that  $T$  is invertible without using your work in subsequent questions as justification. (15 points)

$$T: P_2 \rightarrow \mathbb{C}^3, \quad T(a + bx + cx^2) = \begin{bmatrix} 3a + 2b + 7c \\ 2a + b + 7c \\ -a - b - c \end{bmatrix}$$

Several approaches, simplest is  $n(T) = 0$ . So  $K(T) = ?$

$$T(a + bx + cx^2) = \underline{0}$$

$$\Rightarrow \begin{bmatrix} 3a + 2b + 7c \\ 2a + b + 7c \\ -a - b - c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$  Homogeneous system, only solution  $a = b = c = 0$ .

$$K(T) = \{0 + 0x + 0x^2\} \Rightarrow n(T) = 0 \Rightarrow \text{injective}$$

$$r(T) + n(T) = \dim(P_2) \Rightarrow r(T) + 0 = 3 \Rightarrow r(T) = 3$$

$$\Rightarrow R(T) = \mathbb{C}^3 \Rightarrow T \text{ surjective}$$

By theorem ILTIS,  $T$  is invertible.

4. Using the same  $T$  as at the top of this page, find an explicit formula for  $T^{-1}$ . (25 points)

Pre images of basis vectors for codomain,  $\mathbb{C}^3$ .

$$T^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \{6 - 5x - x^2\}$$

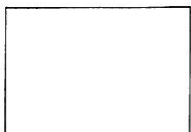
$$T^{-1}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \{-5 - 4x - x^2\}$$

$$T^{-1}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \{7 - x - x^2\}$$

$$\begin{aligned} T^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) &= T^{-1}\left(a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ &= a T^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + b T^{-1}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + c T^{-1}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ &= a(6 - 5x - x^2) + b(-5 - 4x - x^2) + c(7 - x - x^2) \\ &= (6a - 5b + 7c) + (-5a - 4b - c)x + (-a - b - c)x^2 \end{aligned}$$

Answer:

=



5. Suppose that  $A$  is an  $m \times n$  matrix and define the function  $T: \mathbb{C}^m \rightarrow \mathbb{C}^n$  by  $T(\underline{x}) = A\underline{x}$ . Verify that  $T$  meets the definition of a linear transformation. (10 points)

$$\begin{aligned} 1) \quad T(\underline{x}_1 + \underline{x}_2) &= A(\underline{x}_1 + \underline{x}_2) \\ &= A\underline{x}_1 + A\underline{x}_2 \quad \text{MMDA} \\ &= T(\underline{x}_1) + T(\underline{x}_2) \end{aligned}$$

$$\begin{aligned} 2) \quad T(\alpha \underline{x}) &= A(\alpha \underline{x}) \\ &= \alpha(A\underline{x}) \quad \text{MMSMM} \\ &= \alpha T(\underline{x}) \end{aligned}$$

6. Adjust the previous problem so that  $A$  is now an invertible  $n \times n$  matrix. As before, define the function  $T: \mathbb{C}^n \rightarrow \mathbb{C}^n$  by  $T(\underline{x}) = A\underline{x}$ . Verify that  $T$  meets the definition of an invertible linear transformation. (15 points)

Define  $S: \mathbb{C}^n \rightarrow \mathbb{C}^n$  by  $S(\underline{x}) = A^{-1}\underline{x}$  exists!

①  $S$  is a linear transformation by Problem 6.

$$\textcircled{2} \quad (S \circ T)(\underline{x}) = S(T(\underline{x})) = S(A\underline{x}) = A^{-1}(A\underline{x}) = (A^{-1}A)\underline{x} = I\underline{x} = \underline{x}$$

$$\textcircled{3} \quad (T \circ S)(\underline{x}) = T(S(\underline{x})) = T(A^{-1}\underline{x}) = A(A^{-1}\underline{x}) = (AA^{-1})\underline{x} = I\underline{x} = \underline{x}$$

Since both compositions give the identity function,  $S = T^{-1}$ .

7. Suppose that  $S: U \rightarrow V$  is a surjective linear transformation and that  $R = \{\underline{u}_1, \underline{u}_2, \underline{u}_3, \dots, \underline{u}_k\}$  spans the vector space  $U$ . Prove that the set  $T = \{S(\underline{u}_1), S(\underline{u}_2), S(\underline{u}_3), \dots, S(\underline{u}_k)\}$  spans the vector space  $V$ . (15 points)

① Grab an output  $\underline{v} \in V$ .

②  $S$  surjective  $\Rightarrow \underset{\text{input}}{\underline{u}} \in U$  so that  $S(\underline{u}) = \underline{v}$ .

③  $R$  spans  $U \Rightarrow$  scalars  $a_i$  so that  $\underline{u} = a_1\underline{u}_1 + a_2\underline{u}_2 + \dots + a_k\underline{u}_k$ .

$$\begin{aligned} \textcircled{4} \quad \underline{v} &\stackrel{\textcircled{2}}{=} S(\underline{u}) \stackrel{\textcircled{3}}{=} S(a_1\underline{u}_1 + a_2\underline{u}_2 + \dots + a_k\underline{u}_k) \\ &= a_1 S(\underline{u}_1) + a_2 S(\underline{u}_2) + \dots + a_k S(\underline{u}_k) \quad \text{LTLC} \end{aligned}$$

This result says  $T$  spans  $V$ .

