Math 290 Name: Key Dr. Beezer
Exam 5 Spring 2016
Chapters D, E

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. Unless the problem directions specify otherwise, you may use Sage to manipulate matrices and vectors, row-reduce matrices, compute determinants, compute and factor characteristic polynomials, and compute eigenvalues and eigenspaces. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features of something I can't see).

1. Compute the determinant of the following matrix by hand. Sage may not be used to justify your answer and you must show enough work to demonstrate that you know how determinants are computed. (10 points)

$$A = \begin{bmatrix} -2 & -1 & 3 \\ 3 & 1 & -2 \\ 3 & 1 & -1 \end{bmatrix}$$

There is no obvious "best" row or column to expand about, so we will use row 1.

$$\det A = -2\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -2 \\ 3 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= -2\left(1(-1) - (1)(-2)\right) + \left(3(-1) - 3(-2)\right) + 3 \cdot 0$$ ← equal rows

$$= -2(1) + 3 = 1$$

Answer: 1

2. Suppose that A is an 8×8 matrix with $\det(A) = 5$. Determine, with explanation, $\det(2A)$. (15 points)

Each time a row is multiplied by 2, the determinant increases by a factor of 2. With 8 rows, the matrix 2A has 8 rows multiplied by 2. So

$$\det(2A) = 2^8 \det A = 2^8 \cdot 5 = 256 \cdot 5 = 1280$$

Answer: 1280

3. Consider the matrix C. Use Sage only as directed in your justifications of your answers. (40 points)

$$C = \begin{bmatrix} -5 & -2 & -6 & 2 \\ 0 & -13 & 16 & 16 \\ 4 & 1 & 6 & -1 \\ -4 & -13 & 9 & 16 \end{bmatrix}$$

(a) Compute the eigenvalues, and algebraic multiplicities, of C. You may use Sage to find a factored version of the characteristic polynomial.

Sage, factored characteristic polynomial:

$p_C(\lambda) = (\lambda-3)^2(\lambda+1)^2$

Answer:

Eigenvalues: $3, -1$

$\alpha_C(3)=2$, $\alpha_C(-1)=2$

(b) Compute the eigenspaces, and geometric multiplicities, of C. You may use Sage to row-reduce matrices.

$$C-3I_4 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ Theorem BNS $\rightarrow$ $E_C(3) = \left\langle \begin{bmatrix} \\ 1 \\ \\ 0 \end{bmatrix}, \begin{bmatrix} \\ 0 \\ \\ 1 \end{bmatrix} \right\rangle$

$$C+I_4 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem BNS $\rightarrow$ $E_C(-1) = \left\langle \begin{bmatrix} \\ \\ \\ 1 \end{bmatrix} \right\rangle$

Answer:

$E_C(3) = \left\langle \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$; $\gamma_C(3) = 2$

$E_C(-1) = \left\langle \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\rangle$; $\gamma_C(-1) = 1$

(c) Is C diagonalizable? Why or why not?

For $\lambda=-1$, $\alpha_C(-1) = 2 \neq 1 = \gamma_C(-1)$, so by Theorem DMFE, C is not diagonalizable.

4. Are the matrices E and F similar? Why or why not? (15 points)

$$E = \begin{bmatrix} 7 & 3 & 3 \\ -6 & -1 & -6 \\ -2 & -1 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ -3 & -1 & -2 \end{bmatrix}$$

Theorem SMEE: Similar matrices $\Rightarrow$ same eigenvalues

Contrapositive: different eigenvalues $\Rightarrow$ not similar matrices

Using Sage, eigenvalues are:

E: $1, 2, 3$

F: $-1, 1, 2$

so E & F are not similar

5. Construct a 3×3 matrix whose eigenvalues are 1, 2 and 5, and it is not obvious by appearance that these are the eigenvalues of the matrix. Provide enough of a description of your procedure to convince the reader that you have only used the allowed procedures from Sage. (15 points)

The diagonal matrix $D = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 5 \end{bmatrix}$ obviously has eigenvalues $\lambda = 1, 2, 5$ since $p_D(x) = (x-1)(x-2)(x-5)$.

For <u>any</u> nonsingular 3×3 matrix, S, the matrix $S^{-1}DS$ will be similar and by Theorem SMEE will have the same eigenvalues. An example is

$$S = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ -4 & -2 & 5 \end{bmatrix} \Rightarrow B = S^{-1}DS = \begin{bmatrix} -31 & -30 & 12 \\ 32 & 30 & -14 \\ -16 & -14 & 9 \end{bmatrix}$$

almost any random matrix will be nonsingular

6. For a matrix A of size n and a scalar $\alpha \in \mathbf{C}$, determine a formula for $\det(\alpha A)$ in terms of α and $\det(A)$, and give a proof of your formula. Hint: review your work in the second question. (15 points)

As in #2, the matrix αA is the matrix A with n different rows, each multiplied by α. So

$$\det(\alpha A) = \alpha^n \det A.$$