Math 290
Exam 3
Chapter M

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. You may use Sage to form, manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features of something I can't see).

1. Compute the inverse of the matrix B below. Be sure to read the directions above about the use of Sage. (15 points)

$$B = \begin{bmatrix} 1 & -2 & 3 & -3 \\ 1 & -1 & 0 & 1 \\ 3 & -4 & 4 & -4 \\ -3 & 3 & -1 & 1 \end{bmatrix}$$

$$[B \mid I_4] \xrightarrow{\text{RREF}} [I_4 \mid B^{-1}]$$

Theorem CINM

Answer:

$$B^{-1} = \begin{bmatrix} -8 & 0 & 7 & 4 \\ -9 & 0 & 8 & 5 \\ -4 & 1 & 4 & 3 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

2. Let S the set of four vectors that form the columns of the matrix B in the previous question. Is S linearly independent or not? Cite theorems as part of an explanation of your answer that begins with the fact that A has an inverse and which does not simply quote the NMEx series of results. (15 points)

B has an inverse $\Rightarrow$ B nonsingular — Theorem NI

$\Rightarrow$ Columns of B are linearly independent — Theorem NMLIC

3. Consider the matrix A below. In each part of this question compute a linearly independent set whose span equals the requested set of vectors, and which meets the extra conditions of each question. (40 points)

$$A = \begin{bmatrix} -39 & -10 & 76 & -93 & 154 & -296 & 365 & 152 \\ -83 & -21 & 162 & -199 & 328 & -631 & 777 & 324 \\ -29 & -7 & 57 & -71 & 115 & -222 & 272 & 114 \\ 5 & 1 & -10 & 13 & -20 & 39 & -47 & -20 \\ 10 & 3 & -19 & 22 & -39 & 74 & -93 & -38 \end{bmatrix}$$

(a) Column space of A, $\mathcal{C}(A)$. Each vector of your set is a column of A.

Theorem BCS

$D = \{1, 2, 3\}$

$$[A \mid I_5] \Rightarrow \left[\begin{array}{cccccccc|ccccc} 1 & 0 & 0 & -1 & -2 & 2 & -3 & 0 & 0 & 0 & 11 & 38 & 13 \\ 0 & 1 & 0 & -2 & 0 & -1 & -2 & 0 & 0 & 0 & -5 & -19 & -5 \\ 0 & 0 & 1 & -2 & 1 & 3 & 3 & 2 & 0 & 0 & 5 & 17 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

Answer: $\left\{ \begin{bmatrix} -39 \\ -83 \\ -29 \\ 5 \\ 10 \end{bmatrix}, \begin{bmatrix} -10 \\ -21 \\ -7 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 76 \\ 162 \\ 57 \\ -10 \\ -19 \end{bmatrix} \right\}$

(b) Column space of A, $\mathcal{C}(A)$. The first few entries of your vectors have a nice "pattern of zeros and ones" which make the linear independence obvious.

Row-reduce A^T and use Theorem BRS, keep nonzero rows, write as column vectors

Answer: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) Column space of A, $\mathcal{C}(A)$. The set is computed using part of the "J" matrix from extended echelon form.

L = last two rows of J = $\begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 2 \end{bmatrix}$ $F = \{4, 5\}$

$\mathcal{C}(A) = \mathcal{N}(L) = \left\langle \left\{ \begin{bmatrix} \\ \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$

Answer: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(d) Row space of A, $\mathcal{R}(A)$.

Non zero rows of C (in part (a)) expressed as column vectors

Answer: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -2 \\ 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 0 \\ -1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \\ 3 \\ 3 \\ 2 \end{bmatrix} \right\}$

4. For a scalar $\alpha \in \mathbb{C}$ and an $m \times n$ matrix A, prove that $(\alpha A)^t = \alpha A^t$. Provide justification for each step of your proof in a careful style. (15 points)

For $1 \leq i \leq n$, $1 \leq j \leq m$

$$\begin{aligned}
[\underbrace{(\alpha A)^t}_{\text{An } n \times m \text{ matrix}}]_{ij} &= [\alpha A]_{ji} && \text{Defn TM} \\
&= \alpha [A]_{ji} && \text{Defn MSM} \\
&= \alpha [A^t]_{ij} && \text{Defn TM} \\
&= [\alpha A^t]_{ij} && \text{Defn MSM}
\end{aligned}$$

So by Definition ME, $(\alpha A)^t = \alpha A^t$

5. For a scalar $\alpha \in \mathbb{C}$, $m \times n$ matrix A and an $m \times p$ matrix B, prove that $\alpha (AB) = (\alpha A) B$. (15 points)

For $1 \leq i \leq m$, $1 \leq j \leq n$

$$\begin{aligned}
[\alpha(AB)]_{ij} &= \alpha [AB]_{ij} && \text{Defn MSM} \\
&= \alpha \sum_{k=1}^{n} [A]_{ik} [B]_{kj} && \text{Theorem EMP} \\
&= \sum_{k=1}^{n} \alpha [A]_{ik} [B]_{kj} && \text{Property DCN} \\
&= \sum_{k=1}^{n} [\alpha A]_{ik} [B]_{kj} && \text{Defn MSM} \\
&= [(\alpha A) B]_{ij} && \text{Theorem EMP}
\end{aligned}$$

So by Definition ME, $\alpha(AB) = (\alpha A)B$