Math 290 Name: Key Dr. Beezer
Exam 2 Spring 2016
Chapter V

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. You may use Sage to form, manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features of something I can't see).

1. Is the vector $\mathbf{y}$ an element of the span of $T$, $\langle T\rangle$? Explain carefully why, or why not. (15 points)

$$\mathbf{y} = \begin{bmatrix} 7 \\ 36 \\ 17 \\ -4 \end{bmatrix} \qquad T = \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -5 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 1 \\ -3 \end{bmatrix} \right\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

Scalars $\alpha_1, \alpha_2, \alpha_3$ so that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{y}$ ?

Theorem SLSLC $\Rightarrow$ solution to system w/ augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 7 \\ 3 & -2 & 7 & 36 \\ 4 & -5 & 1 & 17 \\ 2 & -4 & -3 & -4 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \alpha_1 = 2,\ \alpha_2 = -1,\ \alpha_3 = 4 \Rightarrow \mathbf{y} \in \langle T\rangle$

2. Write a nontrivial relation of linear dependence on $S$, or explain why no such thing exists. (15 points)

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -1 \\ 8 \end{bmatrix} \right\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

$$[\mathbf{v}_1 | \mathbf{v}_2 | \mathbf{v}_3] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$n = 3 = r$
So by Theorem LIVRN, the set $S$ is linearly independent, so there is <u>no</u> nontrivial relation of linear dependence.

3. Given the matrix $A$, use the appropriate theorem to find a linearly independent set $R$ so that the span of $R$ is the null space of $A$, $\langle R\rangle = \mathcal{N}(A)$. (20 points)

$$A = \begin{bmatrix} 1 & 3 & 4 & -7 & -5 & 8 \\ 0 & 1 & 2 & -4 & -2 & 3 \\ -1 & -2 & -2 & 3 & 4 & -7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -2 & 5 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad D = \{1, 2, 5\} \quad F = \{3, 4, 6\}$$

$$F \Rightarrow R = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(linearly independent)

"empty" in slots 1,2,5

Theorem BNS $\Rightarrow \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\} = R$

Answer:

4. Use the appropriate theorem to find a set $K$ that is (1) a subset of $S$, (2) linearly independent, and (3) the span of $K$ equals the span of $S$, $\langle K\rangle = \langle S\rangle$. (20 points)

$$S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -7 \\ 7 \end{bmatrix} \right\}$$

By Theorem BS, make these vectors the columns of a matrix, and row-reduce

$$[\mathbf{v}_1 | \mathbf{v}_2 | \mathbf{v}_3 | \mathbf{v}_4 | \mathbf{v}_5] \xrightarrow{\text{RREF}}$$

$$\begin{bmatrix} 1 & 0 & 6 & 0 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Pivot columns $D = \{1, 2, 4\}$

so $K = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \right\}$

Answer:

5. For scalars $\alpha, \beta$ and vectors $\mathbf{u} \in \mathbb{C}^n$, prove that $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$. Provide justification for each step of your proof in a careful style. (15 points)

For $1 \le i \le n$,

$$
\begin{aligned}
[(\alpha\beta)\mathbf{u}]_i &= (\alpha\beta)[\mathbf{u}]_i && \text{Defn CVSM} \\
&= \alpha(\beta[\mathbf{u}]_i) && \text{Property MACN} \\
&= \alpha[\beta\mathbf{u}]_i && \text{Defn CVSM} \\
&= [\alpha(\beta\mathbf{u})]_i && \text{Defn CVSM}
\end{aligned}
$$

Scalar equalities

So by Defn CVE, $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$.

vector equality

6. For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ prove that the inner product is additive, $\langle \mathbf{u} + \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$. (This is a theorem in the book, so do more than just quoting that theorem.) (15 points)

$$
\begin{aligned}
\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle &= \sum_{i=1}^{n} \overline{[\mathbf{u}+\mathbf{v}]_i}[\mathbf{w}]_i && \text{Defn IP} \\
&= \sum_{i=1}^{n} \left(\overline{[\mathbf{u}]_i + [\mathbf{v}]_i}\right)[\mathbf{w}]_i && \text{Defn CVA} \\
&= \sum_{i=1}^{n} \left(\overline{[\mathbf{u}]_i} + \overline{[\mathbf{v}]_i}\right)[\mathbf{w}]_i && \text{Theorem CCRA} \\
&= \sum_{i=1}^{n} \overline{[\mathbf{u}]_i}[\mathbf{w}]_i + \overline{[\mathbf{v}]_i}[\mathbf{w}]_i && \text{Prop DCN} \\
&= \sum_{i=1}^{n} \overline{[\mathbf{u}]_i}[\mathbf{w}]_i + \sum_{i=1}^{n} \overline{[\mathbf{v}]_i}[\mathbf{w}]_i && \text{Prop. CACN} \\
&= \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle && \text{Defn IP}
\end{aligned}
$$