# The Fundamental Algebraic Group of Topological Spaces

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## 1 Abstract

In the study of topology, we are often interested in understanding and classifying the internal structure of topological spaces. Algebraic topology is the application of abstract algebra to topology in order to further identify the structure of topological spaces by developing a correspondence between topological spaces and certain groups called homotopy groups. In this paper, we will examine the construction and nature of the first homotopy group, which is more commonly known as the fundamental group of a topological space.

This paper is intended for undergraduate students of group theory with little to no background in point-set topology and will provide an introduction to algebraic topology and homotopy groups. We will first briefly cover the basics of point-set topology, then use these concepts to facilitate a rigorous study of the construction of the fundamental group. Afterwards, we will examine various proofs describing the nature of the fundamental group and demonstrate basic methods of calculation of the fundamental group of a topological space.

## 2 Basic Topology

In order to properly construct the fundamental group, we require a basic understanding of point-set topology.

**Definition 2.1:** A *topological space* is any nonempty set $X$ paired with a collection of subsets of $X$ known as *open sets* satisfying:

($i$) $X$ and $\varnothing$ are contained in the collection of open sets.
($ii$) The finite or infinite union of any open sets is itself an open set.
($iii$) The finite intersection of any open sets is itself an open set.

■

The collection of open sets of a topological space is called the *topology* on a set $X$. Given a set $X$ and a collection of subsets forming a topology on $X$, we will commonly refer to the topological space as "the space $X$" and refer to elements of the topology on $X$ as "the open sets of $X$".

**Example 2.2:** We can define a topology on $\mathbb{R}^n$ by letting the collection of open sets be the set of all possible arbitrary unions and finite intersections of subsets of $\mathbb{R}^n$ of the form $U = \{x \in \mathbb{R}^n : d(x,y) < \varepsilon\}$ for any $y \in \mathbb{R}^n, \varepsilon > 0$, where $d(x,y)$ is the Euclidean distance between the points $x$ and $y$. For example, $(-1,1) = \{x \in \mathbb{R} : d(x,0) < 1\}$ is an open set in $\mathbb{R}^1$.

Given this topology, we can define a topology on $I = [0,1] \subset \mathbb{R}^1$ by letting a set $V \subseteq I$ be open if there exists an open set $U \subseteq \mathbb{R}^1$ such that $V = I \cap U$. For example, $(\frac{1}{4}, \frac{3}{4}) = \{x \in I : d(x, \frac{1}{2}) < \frac{1}{4}\}$ and $[0, \frac{1}{4}) \cup (\frac{3}{4}, 1] = \{x \in I : d(x,0) < \frac{1}{4}\} \cup \{x \in I : d(x,1) < \frac{1}{4}\}$ are both open sets in this topology. ■

When we constructed the topological space $I$ as a subset of another topological space in the manner above, we built what is called the *relative topology* on $I$ as a subset of $\mathbb{R}^1$.

Later on, the spaces $\mathbb{R}^n$ and $I = [0,1]$ of Example 2.2 will be instrumental in constructing the fundamental group. For this reason, when we later on make reference to either space we will be referring to the topological spaces as described above unless we explicitly state otherwise.

**Definition 2.3:** Given two topological spaces $X$ and $Y$, the *product topology* of $X$ and $Y$ is the set $X \times Y$ with a topology consisting of all possible arbitrary unions and finite intersections of subsets of the form $U \times V$, where $U$ is open in $X$ and $V$ is open in $Y$. The topological space $X \times Y$ is referred to as the *product space.* ■

**Definition 2.4:** Given two topological spaces $X$ and $Y$, a mapping $f : X \to Y$ is *continuous* if $f^{-1}(V)$ is an open set in $X$ for every open set $V \subseteq Y$. ■

While we will not prove it here, when considering the topological space $\mathbb{R}^n$, this topological definition of continuity is identical to the more common epsilon-delta definition of continuity. While we will almost exclusively use the topological definition of continuity in this paper, this fact may help the reader better understand the relationship between continuous functions and open sets.

**Propostion 2.5:** If $X, Y, Z$ are topological spaces and $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $g \circ f$ is a continuous function.

*Proof.* Suppose $U$ is an open set in $Z$. Since $g$ is a continuous function, $g^{-1}(U)$ is open in $Y$. Since $f$ is continuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in $X$. Therefore, $g \circ f$ is continuous. □

The following theorem will later on be of critical importance to our construction of the fundamental group.

**Theorem 2.6: (Map Gluing Theorem)** Suppose $A, B$ are subsets of a topological space $X$ such that $A \cup B = X$ and such that there exist open sets $U, V \subseteq X$ where $A = X \backslash U$ and $B = X \backslash V$. If $Y$ is a topological space and $f : X \to Y$ and $g : X \to Y$ are well-defined continuous functions such that $f|_{A \cap B} = g|_{A \cap B}$, then the function $h : X \to Y$ described by

$$h(x) = \begin{cases} f(x) & \text{for } x \in A \\ g(x) & \text{for } x \in B \end{cases}$$

is continuous.

*Proof.* Because $f|_{A \cap B} = g|_{A \cap B}$ and $f$ and $g$ are both well-defined, $h$ is clearly a well-defined function. Let $W$ be an open set in $Y$. Then, $f^{-1}(W)$ and $g^{-1}(W)$ are open in $X$ since $f$ and $g$ are continuous. Furthermore, $f^{-1}(Y \backslash W) = X \backslash f^{-1}(W)$ and $g^{-1}(Y \backslash W) = X \backslash g^{-1}(W)$. Since $A = X \backslash U$ and $B = X \backslash V$, therefore $f^{-1}(Y \backslash W) \cap A = X \backslash (f^{-1}(W) \cap U)$ and $g^{-1}(Y \backslash W) \cap B = X \backslash (g^{-1}(W) \cap V)$

Furthermore, it is clear that

$$X \backslash h^{-1}(W) = h^{-1}(Y \backslash W) = (f^{-1}(Y \backslash W) \cap A) \cup (g^{-1}(Y \backslash W) \cap B)$$

$$= X \backslash (f^{-1}(W) \cap U) \cup X \backslash (g^{-1}(W) \cap V) = X \backslash \big( (f^{-1}(W) \cup U) \cup (g^{-1}(W) \cup V) \big)$$

This implies that $h^{-1}(W) = (f^{-1}(W) \cup U) \cup (g^{-1}(W) \cup V)$. Since the finite union and intersection of open sets is open, $h^{-1}(W)$ is open in $X$ and $h$ is therefore continuous. □

# 3 Construction of the Fundamental Group

With some basic topological definitions and tools now available to us, we begin our study of the fundamental group. We will first demonstrate the method of construction of the fundamental group, then prove that it is indeed a group.

**Definition 3.1:** If $X$ is a topological space and $f : I \to X$ is a continuous function, then $f$ is a *path* in $X$. Given a path $f$, $f(0)$ and $f(1)$ are respectively known as the *initial point* and *terminal point* of the path. If $f$ is a path such that $f(0) = f(1)$, then $f$ is a *loop* based at the point $f(0)$. ■

Note that in a topological space $X$, it is not necessarily true that there exists a path from any one point to another. For example, if we let $I \backslash \{\frac{1}{2}\}$ have the relative topology as a subset of $I$, then there does not exist a continuous path from $\frac{1}{4}$ to $\frac{3}{4}$. If, however, it is true that any pair of points in a space $X$ have a continuous path connecting them, then the space $X$ is called *path connected*.

**Definition 3.2:** Given topological spaces $X$ and $Y$ and continuous functions $f, g : X \to Y$, the functions $f$ and $g$ are *homotopic* to each other if there exists a continuous function $H : X \times I \to Y$ such that $H_0 = H(x, 0) = f(x)$ and $H_1 = H(x, 1) = g(x)$. In this scenario, $H$ is called a *homotopy* from $f$ to $g$. ■

Intuitively, the concept of two paths being homotopic coincides with the idea of each path being able to be continuously deformed into the other. Given a homotopy $H$ from a path $f$ to a path $g$, when evaluating at $H(i, t)$, $i, t \in I$, it may help to think of $t$ as being the variable for time as $f$ continuously deforms into $g$. At time zero the function $H$ copies $f$, and at time one it copies $g$. At every time between zero and one, $H$ traces some path in $X$ between $f$ and $g$, since $H$ is continuous.

With these definitions, we can begin our construction of the fundamental group. Suppose $X$ is a topological space and let $b \in X$. Let $L$ be the set of all loops in $X$ based at the point $b$. Define a relation $\sim$ on $L$ where for any two loops $f, g \in L$, $f \sim g$ if and only if $f$ is homotopic to $g$.

**Theorem 3.3:** $\sim$ defines an equivalence relation on $L$.

*Proof.* Let $f, g, h \in L$.

*Reflexive:* Let $H : I \times I \to X$ by $H(i, t) = f(i)$ for all $i, t \in I$. Since $f$ is continuous, so is $H$. $H$ then clearly describes a homotopy from $f$ to $f$. Thus, $f \sim f$.

*Symmetric:* Suppose that $f \sim g$. Then there exists some homotopy $H$ from $f$ to $g$. Let $H' : I \times I \to X$ by $H'(i, t) = H(i, 1 - t)$ for all $i, t \in I$. Then, $H'_0 = g(i)$ and $H'_1 = f(i)$. Furthermore, since $1 - t$ is a continuous function and because composition of continuous functions is continuous by Proposition 2.5, $H'$ is a continuous function. Therefore, $g \sim f$.

*Transitive:* Suppose that $f$ is equivalent to $g$ by a homotopy $F$ and that $g$ is equivalent to $h$ by a homotopy $G$. Let $H : I \times I \to X$ by

$$H(i, t) = \begin{cases} F(i, 2t) & \text{for } t \in [0, \frac{1}{2}] \\ G(i, 2t - 1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

for all $i \in I$. Since $2t$ and $2t - 1$ and $F$ and $G$ are all continuous functions, therefore $F(i, 2t)$ and $G(i, 2t - 1)$ are continuous functions since the composition of continuous functions is continuous. By the Map Gluing Theorem, $H$ is therefore a continuous function, and so $f \sim h$. □

Now that we have proven $\sim$ is an equivalence relation, we can form $L/\sim$ as the set of equivalence classes of $L$ under $\sim$. This will become the set of the fundamental group. At this point, there remains only one more component to forming the fundamental group: a multiplicative operation.

Given two loops $f, g : I \to X$ in $L$, define an operation $*$ on $L$ by

$$(f * g)(t) = \begin{cases} f(2t) & \text{for } t \in [0, \frac{1}{2}] \\ g(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Note that because $2t$ and $2t-1$ are continuous, therefore $f(2t)$ and $g(2t-1)$ are continuous by Proposition 2.3. By the Map Gluing Theorem, $f * g$ is continuous and is therefore a path in $L$ since it is clearly based at the point $b$.

Intuitively, $f * g$ is a path which first follows the path of $f$, then follows the path of $g$. To meet our restriction to $[0, 1]$ however, we follow the paths of $f$ and $g$ twice as fast as we did individually.

Given the operation $*$ on $L$, we can extend it to $L/\sim$ by defining an operation on $L/\sim$ as: $[f][g] = [f * g]$ for all $[f], [g] \in L/\sim$. The set $L/\sim$ combined with this operation is the *fundamental group* of $X$ based at $b$, and is denoted as $\pi_1(X, b)$.

**Theorem 3.4:** Given a topological space $X$ and a base point $b \in X$, the fundamental group $\pi_1(X, b)$ forms a group under the operation $[f][g] = [f * g]$ for all $[f], [g] \in \pi_1(X, b)$.

We shall prove this result through a series of lemmas.

**Lemma 3.5:** Given a fundamental group $\pi_1(X, b)$, the operation $[f][g] = [f * g]$ for all $[f], [g] \in \pi_1(X, b)$ is well-defined.

*Proof.* Suppose that $[f_1] = [f_2]$ and $[g_1] = [g_2]$, for some $[f_1], [f_2], [g_1], [g_2] \in \pi_1(X, b)$. Then there must exist homotopies $F, G : I \times I \to X$ taking $f_1$ to $f_2$ and $g_1$ to $g_2$ respectively. To show that $[f_1][g_1] = [f_1 * g_1] = [f_2 * g_2] = [f_2][g_2]$, we shall find a homotopy taking $f_1 * g_1$ to $f_2 * g_2$. Let $H : I \times I \to X$ by

$$H(i, t) = \begin{cases} F(2i, 2t) & \text{for } t \in [0, \frac{1}{2}], i \in [0, \frac{1}{2}] \\ g_1(2i-1) & \text{for } t \in [0, \frac{1}{2}], i \in [\frac{1}{2}, 1] \\ \\ f_2(2i) & \text{for } t \in [\frac{1}{2}, 1], i \in [0, \frac{1}{2}] \\ G(2i-1, 2t-1) & \text{for } t \in [\frac{1}{2}, 1], i \in [\frac{1}{2}, 1] \end{cases}$$

As we have seen several times now, we can use the fact that the composition of continuous functions is continuous as well as the Map Gluing Theorem to arrive at the conclusion that $H$ is continuous and therefore describes a homotopy from $f_1 * g_1$ to $f_2 * g_2$. Therefore, the operation on $\pi_1(X, b)$ is well-defined. $\square$

This homotopy may seem cryptic at first glance, but careful examination reveals that we are deforming $f_1 * g_1$ into $f_2 * g_2$ by first using the homotopy $F$ to deform $f_1$ into $f_2$ while leaving $g_1$ as is, then using $G$ to deform $g_1$ into $g_2$ while leaving $f_2$ untouched. The spacing between the cases serves to visually highlight the difference between deforming $f_1$ into $f_2$ and deforming $g_1$ into $g_2$.

**Lemma 3.6:** Given a fundamental group $\pi_1(X, b)$, the operation $[f][g] = [f * g]$ for all $[f], [g] \in \pi_1(X, b)$ is associative.

*Proof.* Let $\pi_1(X, b)$ be a fundamental group and suppose any $[f], [g], [h] \in \pi_1(X, b)$. To show that $[f]([g][h]) = ([f][g])[h]$, we shall find a homotopy from $f * (g * h)$ to $(f * g) * h$. Define

$$H(i, t) = \begin{cases} f(\frac{4i}{2-t}) & \text{for } 0 \leq i \leq \frac{2-t}{4} \\ g(4i - 2 + t) & \text{for } \frac{2-t}{4} \leq i \leq \frac{3-t}{4} \\ h(\frac{4i-3+t}{1+t}) & \text{for } \frac{3-t}{4} \leq s \leq 1 \end{cases}$$

Again using the Map Gluing Theorem and the fact that the composition of continuous functions is continuous, $H$ is therefore continuous and describes a homotopy from $f*(g*h)$ to $(f*g)*h$. Therefore, $[f]([g][h]) = ([f][g])[h]$, making the operation associative. $\square$

Despite this homotopy being even more cryptic than the last, it turns out to be relatively simple. Since we do not change the order we perform $f, g$, and $h$, all we need to is change the amount of time we spend on each function. Starting at $f*(g*h)$, we trace $f$ for the first half of time, then trace $g$ for a quarter of the time and we finally trace $h$ for the final quarter of time. The homotopy then slowly shifts the time we spend on each function in relation to $t$, ending with a path where we trace $f$ for only the first quarter of time, $g$ for the second quarter, and we finish by tracing $h$ for the second half. The reader should verify that this is indeed true and that $H_0 = f*(g*h)$ and $H_1 = (f*g)*h$.

**Lemma 3.7:** Given a fundamental group $\pi_1(X,b)$, if $c : I \to X$ is the constant map $c(i) = b$ for all $i \in I$, then $[f][c] = [f]$ and $[c][f] = [f]$ for all $[f] \in \pi_1(X,b)$.

*Proof.* To prove that $[f][c] = [f]$ for any $[f] \in \pi_1(X,b)$, define

$$H(i,t) = \begin{cases} b & \text{for } 0 \le t \le -2i+1 \\ f(\frac{2i+t-1}{t+1}) & \text{for } -2i+1 \le t \le 1 \end{cases}$$

As per usual, we use the Map Gluing Theorem and the continuity of the composition of continuous functions to prove that $H$ is continuous and therefore describes a homotopy from $f*c$ to $f$. A similar function will demonstrate that $[c][f] = [f]$. $\square$

To intuitively describe this homotopy, we begin with a path which first traces $f$ half the time then $c$ the remainder of the time. As $t$ increases, the homotopy steadily increases the amount of time we spend tracing $f$ and decreases the amount of time we spend tracing $c$ until the path traces $f$ the entire time. Again, the reader should verify this. The reader should also recognize that $[c]$ acts as the identity element $e$ of $\pi_1(X,b)$.

To prove the existence of inverse elements, we require some new notation. If $f : I \to X$ is a path, let $f^r : I \to X$ be the path defined by $f^r(i) = f(1-i)$ for all $i \in I$. Note that $f^r$ is essentially the path $f$ in reverse.

**Lemma 3.8:** Given a fundamental group $\pi_1(X,b)$, if we suppose any element $[f] \in \pi_1(X,b)$, then $[f][f^r] = [c] = [f^r][f]$, where $c : I \to X$ is the constant function mapping to $b$.

*Proof.* We shall find a homotopy such that $f*f^r \sim c$. Define $H : I \times I \to X$ by

$$H(i,t) = \begin{cases} b & \text{for } 0 \le i \le \frac{t}{2} \\ f(2i-t) & \text{for } \frac{t}{2} \le i \le \frac{1}{2} \\ f^r(2i+t-1) & \text{for } \frac{1}{2} \le i \le \frac{2-t}{2} \\ b & \text{for } \frac{2-t}{2} \le i \le 1 \end{cases}$$

Once more, the Map Gluing Theorem and the continuity of the composition of continuous functions proves that $H$ is continuous and therefore describes a homotopy from $f*f^r$ to $c$. A similar function will provide a homotopy from $f^r*f$ to $c$. $\square$

Intuitively, this homotopy describes how the path $f*f^r$, which traces $f$ then retraces its steps with $f^r$, is essentially the same as having never left the base point at all. As $t$ increases from 0 to 1 we steadily decrease the distance traveled by $f$ and retraced by $f^r$ (decreasing the time spent on them) and increase the amount of time we spend waiting at $c$, which we spend waiting before we begin $f$ and after we finish with $f^r$. Eventually, we end up spending all our time at $c$ at $t=1$, and travel no distance along $f$ or $f^r$. As usual, the reader should verify this result, as well as recognize that $[f^r] = [f]^{-1}$ in $\pi_1(X,b)$.

At this point, we have proven enough results about the fundamental group that we can confirm Theorem 3.4: given a topological space $X$ and a base point $b \in X$, $\pi_1(X, b)$ is indeed a group. At this point, we can begin to use the topology we know so far to explore the nature of the fundamental group.

**Example 3.9:** If we let $S^1 = \{(x, y) \in \mathbb{R}^2 : d((x, y), (0, 0)) = 1\}$, we can form a topological space on $S^1$ as the relative topology as a subset of $\mathbb{R}^2$. Then we can pick the point $(1, 0) \in S^1$ and form the fundamental group $\pi_1(S^1, (1, 0))$.

To understand this group on an intuitive level, recognize that the simplest non-trivial loop in $\pi_1(S^1, (1, 0))$ is a loop wrapped all the way around $S^1$ once, since this loop clearly cannot be continuously deformed into the identity of $\pi_1(S^1, (1, 0))$: the constant loop based at the point $(1, 0)$. Furthermore, if we wrap the loop in the opposite direction we "undo" the previous loop and end up with the identity. If we wrap a loop multiple times in the same direction, we create a unique element in $\pi_1(S^1, (1, 0))$ with each wrap around since we cannot continuously deform $n$ wrappings into any fewer, or more wrappings. Additionally, if we first wrap a loop $n$ times around $S^1$ then wrap it $m$ times around in the same direction, it is as if we wrapped a loop $n + m$ times around. At this point, it should be clear that at least intuitively, $\pi_1(S^1, (1, 0))$ is isomorphic to the group $\mathbb{Z}$ under addition, which in fact it is.

This is a critical result in algebraic topology, and while the proof that $\pi_1(S^1, (1, 0)) \cong \mathbb{Z}$ is beyond the scope of this paper, we will regularly cite this fact to create various examples, as well as illustrate underlying concepts of important theorems. ■

# 4 Nature of the Fundamental Group

**Lemma 4.1:** If $X$ and $Y$ are topological spaces, $f, g : I \to X$ are loops in $X$ based at some point $b$, and $h : X \to Y$ is continuous, then $h \circ (f * g) = (h \circ f) * (h \circ g)$.

*Proof.* By definition for all $i \in I$, if $i \leq \frac{1}{2}$ then $h \circ (f * g)(i) = h \circ (f(2i))$ and $(h \circ f) * (h \circ g)(i) = h \circ f(2i)$. Furthermore, if $i \geq \frac{1}{2}$, then $h \circ (f * g)(i) = h \circ (g(2i))$ and $(h \circ f) * (h \circ g)(i) = h \circ g(2i)$. Therefore, $h \circ (f * g) = (h \circ f) * (h \circ g)$. Additionally, since $f, g, h$ are all continuous, $h \circ (f * g) = (h \circ f) * (h \circ g)$ is therefore a loop in $Y$ based at $h(b)$. □

**Theorem 4.2** If $\pi_1(X, b)$ and $\pi_1(Y, v)$ are the fundamental groups of $X$ and $Y$ based at $b$ and $v$ respectively, then $\pi_1(X \times Y, (b, v)) \cong \pi_1(X, b) \times \pi_1(Y, v)$.

*Proof.* Let $X$ and $Y$ be topological spaces and let $b \in X$ and $v \in Y$. Define $p_x : X \times Y \to X$ by $p_x(\alpha, \beta) = \alpha$ for all $(\alpha, \beta) \in X \times Y$. Then $p_x$ is continuous, since if we let $U \subseteq X$ be open in $X$, then $p_x^{-1}(U) = U \times Y$, which is open in $X \times Y$ by definition of the product topology. Define $p_y : X \times Y \to Y$ in a similar manner.

Define $\phi : \pi_1(X \times Y, (b, v)) \to \pi_1(X, b) \times p_1(Y, v)$ by $\phi([f]) = ([p_x \circ f], [p_y \circ f])$ for all $[f] \in \pi_1(X \times Y, (b, v))$. By the nature of $p_x$ and $p_y$, $\phi$ is trivially shown to be well-defined. We claim that $\phi$ is an isomorphism.

To prove that $\phi$ is a homomorphism, suppose any $[f], [g] \in \pi_1(X \times Y, (b, v))$. Then,

$$\phi([f][g]) = \phi([f * g]) = ([p_x \circ (f * g)], [p_y \circ (f * g)])$$

By Lemma 4.1, we have:

$$= ([p_x \circ f][p_x \circ g], [p_y \circ f][p_y \circ g]) = ([p_x \circ f], [p_y \circ f]) ([p_x \circ g], [p_y \circ g]) = \phi([f])\phi([g])$$

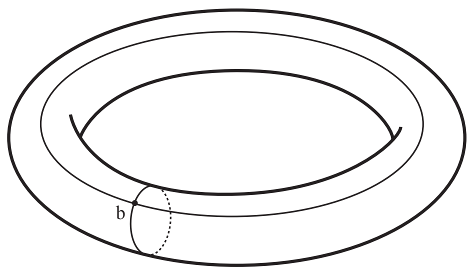
To prove that $\phi$ is injective, suppose any $[f], [g] \in \pi_1(X \times Y, (b, v))$ such that $[f] = [g]$. Let $H : I \times I \to X \times Y$ be a homotopy from $f$ to $g$. Define $F = p_x \circ H$ and $G = p_y \circ H$. Since $H$ and $p_x$ and $p_y$ are continuous, therefore $F$ and $G$ are continuous. Furthermore, $F_0 = p_x \circ f$, $F_1 = p_x \circ g$, $G_0 = p_y \circ f$, and $G_1 = p_y \circ g$. Therefore, $[p_x \circ f] = [p_x \circ g]$ and $[p_y \circ f] = [p_y \circ g]$, and so $\phi([f]) = \phi([g])$.

To prove that $\phi$ is surjective, suppose any $([f],[g]) \in \pi_1(X,b) \times \pi_1(Y,v)$. Define $h : I \to X \times Y$ by $h(i) = (f(i), g(i))$ for all $i \in I$. Clearly, $p_x \circ h = f$ and $p_y \circ h = g$. Furthermore, the continuity of $f$ and $g$ implies that $h$ is continuous, and so $\phi([h]) = ([p_x \circ h], [p_y \circ h]) = ([f],[g])$.

□

This theorem frequently provides us with a tool to simplify the calculation of the fundamental group of a given space, as we shall demonstrate in the next example.

**Example 4.3:** The surface of a torus can be expressed as the product topology $S^1 \times S^1$. Since we've already established that $\pi_1(S^1,(1,0)) \cong \mathbb{Z}$, Theorem 4.2 proves that the fundamental group of the surface of a torus based at the point $b = ((1,0),(1,0))$ is therefore $\mathbb{Z} \times \mathbb{Z}$. Intuitively, this can understood as having two different kinds of loops generating the rest of the group with each loop becoming "snagged" on either the first $S^1$ in the product topology or the second, as is shown in the below diagram. ■




**Theorem 4.4** If $X$ is a path connected topological space and $x \in X$, then $\pi_1(X,x) \cong \pi_1(X,y)$ for all $y \in X$.

*Proof.* Let $X$ be a path connected space and suppose $x, y \in X$. By the definition of path connectedness, there exists some path $f : I \to X$ such that $f(0) = x$ and $f(1) = y$. Define $\phi : \pi_1(X,x) \to \pi_1(X,y)$ by $\phi([g]) = [f * g * f^r]$ for all $[g] \in \pi_1(X,x)$. We claim that $\phi$ is an isomorphism.

To prove that $\phi$ is well-defined, let $[g],[h] \in \pi_1(X,x)$ such that $[g] = [h]$. Let $H$ be a homotopy from $g$ to $h$ and let $F : I \times I \to X$ by $F(i,t) = f * H_t * f^r(i)$. Clearly, $F_0 = f * g * f^r$ and $F_1 = f * h * f^r$. Furthermore, since $H$ and $g$ and $f$ are all continuous, $F$ is continuous. Thus, $[f * g * f^r] = [f * h * f^r]$ and so, $\phi([g]) = [f * g * f^r] = [f * h * f^r] = \phi([h])$.

To prove that $\phi$ is a homomorphism, suppose any $[g],[h] \in \pi_1(X,x)$. Then,

$$\phi([g][h]) = \phi([g * h]) = [f * g * h * f^r] = [f * g * f^r * f * h * f^r] = [f * g * f^r][f * h * f^r] = \phi([g])\phi([h])$$

To prove $\phi$ is bijective, define $\eta : \pi_1(X,y) \to \pi_1(X,x)$ by $\eta([g]) = [f^r * g * f]$ for all $[g] \in \pi_1(X,y)$. Then for all $[g] \in \pi_1(X,x)$,

$$\eta(\phi([g])) = \eta([f * g * f^r]) = [f^r * f * g * f^r * f] = [g]$$

and for all $[g] \in \pi_1(X,y)$,

$$\phi(\eta([g])) = \phi([f^r * g * f]) = [f * f^r * g * f^r * f] = [g]$$

Therefore, $\phi^{-1} = \eta$ and $\phi$ thus describes an isomorphism from $\pi_1(X,x)$ to $\pi_1(X,y)$. □

With this theorem in mind, if $X$ is a path connected space we will adopt the notation of $\pi_1(X)$ to represent the fundamental group of $X$, since the choice of a base point is irrelevant.

In addition to providing a better understanding of the underlying concepts of path connected spaces such as $S^1$ or the torus, this theorem also provides us the ability to pick specific base points when calculating the fundamental group of a path connected space. This will later on allow us to take advantage of powerful theorems which can only be applied in the context of specific base points.

**Theorem 4.5:** Let $X$ be a topological space and $A \subseteq X$ be a subset of $X$ with the relative topology such that there exists a continuous map $r : X \to A$ where $r|_A = id_A$. Let $i_A : A \hookrightarrow X$ be the inclusion map defined by $i_A(a) = a$ for all $a \in A$. If there exists a continuous homotopy $H : X \times I \to X$ such that $H_0 = id_X$ and $H_1 = i_A \circ r$, then if $a \in A$ we have: $\pi_1(X, a) \cong \pi_1(A, a)$.

*Proof.* Let $a \in A$ and let $\phi : \pi_1(X, a) \to \pi_1(A, a)$ by $\phi([\alpha]) = [r \circ \alpha]$ for all $[\alpha] \in \pi_1(X, a)$. Since $r$ is continuous, it is straightforward to prove that $\phi$ is a well-defined map. To prove that $\phi$ is a homomorphism, suppose any $[\alpha], [\beta] \in \pi_1(X, a)$. Then,

$$\phi([\alpha][\beta]) = \phi([\alpha * \beta]) = [r \circ (\alpha * \beta)]$$

Since $r$ is continuous, by Lemma 4.1:

$$= [(r \circ \alpha) * (r \circ \beta)] = [r \circ \alpha][r \circ \beta] = \phi([\alpha])\phi([\beta])$$

To prove that $\phi$ is surjective, let $\eta : \pi_1(A, a) \to \pi_1(X, a)$ by $\eta([\alpha]) = [i_A \circ \alpha]$ for all $[\alpha] \in \pi_1(X, a)$. Then if we let $[\alpha] \in \pi_1(A, a)$,

$$\phi(\eta([\alpha])) = \phi([i_A \circ \alpha]) = [(r \circ i_A) \circ \alpha] = [\alpha]$$

since $r \circ i_A = r|_A = id_A$. Therefore, $\eta \circ \phi = id_{\pi_1(A,a)}$, and so $\phi$ is surjective.

Furthermore, if we let $[\alpha] \in \pi_1(X, a)$, we have

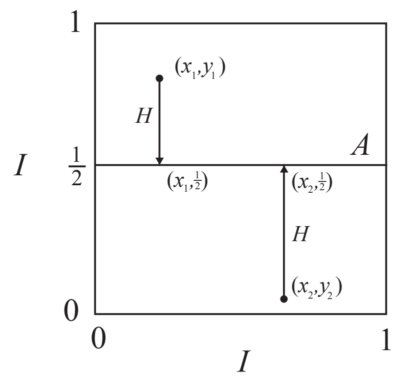
$$\eta(\phi([\alpha])) = \eta([r \circ \alpha]) = [i_A \circ r \circ \alpha] = [H_1 \circ \alpha] = [\alpha]$$

since $H$ is a homotopy from $id_X$ to $i_A \circ r$. Therefore, $\phi \circ \eta = id_{\pi_1(X,a)}$ and so $\phi$ is injective. Therefore, $\phi$ is an isomorphism. □

This homotopy is known as a *deformation retract.* Intuitively, we are continuously deforming $X$ into a subset $A$ in such a way that every path in $X$ deforms into its homotopic equivalent in $A$. If such a subset exists in $X$, then that subset must essentially be homotopically equivalent to $X$, resulting in the two spaces having the same fundamental group.

**Example 4.6:** It is common knowledge that a Mobius strip can be created by taking a strip of paper, twisting one end a half-turn, then gluing the ends together. Topologically, we create this structure by taking the product space $I \times I$ and defining an equivalence relation $\sim$ where $(0, x) \sim (1, 1-x)$ for all $(0, x), (1, 1-x) \in I \times I$ and $(x, y) \sim (x, y)$ for all other points $(x, y)$. This is the topological equivalent of gluing the ends together after a half-turn. Let $X$ be the set of all equivalence classes of $I \times I$ and define a topology on $X$ by letting $U \subseteq X$ be open in $X$ if and only if $q^{-1}(U)$ is open in $I \times I$, where $q : I \times I \to X$ is a map defined by $q(x, y) = [(x, y)]$. The reader should verify this is indeed a Mobius strip.

Let $A = \{[(x, \frac{1}{2})] : x \in I\} \subset X$. It should be clear that because $(0, \frac{1}{2}) \sim (1, \frac{1}{2})$, we have taken a line and joined the endpoints together. This makes $A$ roughly equivalent to $S^1$, at least up to their fundamental groups (we will later see how to do this more rigorously). Therefore, $\pi_1(A) \cong \pi_1(S^1) \cong \mathbb{Z}$. Using this fact, define a function $H : X \times I \to X$ by $H([(x, y)], t) = [(x, y(1-t) + \frac{t}{2})]$ for all $[(x, y)] \in X$. Intuitively, this map is a continuous function which slides every point on the Mobius strip to $A$, as is shown in the diagram below. It is not difficult to prove this function is well-defined, and the use of Lemma 4.1 and the fact that composition of continuous functions is continuous proves that $H$ is continuous and therefore a homotopy from $X$ to $A$. It should furthermore be clear that $H$ satisfies the homotopy described in Theorem 4.4. Therefore, $\pi_1(X) \cong \pi_1(A) \cong \mathbb{Z}$. ■




**Definition 4.7:** Suppose that $X$ and $Y$ are topological spaces. If $f : X \to Y$ is a bijective and continuous function such that $f^{-1}$ is continuous, then the spaces $X$ and $Y$ are *homeomorphic* and $f$ describes a *homeomorphism* between them. ■

This concept is exceedingly important in topology and is one of the critical motivators behind the study of topological spaces. Since the open sets of a topological space define the internal structure of the space much like how the relationships between elements define the structure of a group, a homeomorphism between spaces is the topological equivalent of an isomorphism between groups. Naturally, since the internal structure of the fundamental group of a space is dependent on the internal structure of that space, we question whether an isomorphism between fundamental groups of spaces is related to homeomorphisms between those spaces. This next theorem will help answer this question.

**Theorem 4.8:** Suppose $X$ and $Y$ are topological spaces such that $f : X \to Y$ is a homeomorphism. Then, $\pi_1(X, x) \cong \pi_1(Y, f(x))$ for any $x \in X$.

*Proof.* Let $x \in X$ and define $\phi : \pi_1(X, x) \to \pi_1(Y, f(x))$ by $\phi([g]) = [f \circ g]$ for all $[g] \in \pi_1(X, x)$. Since $f$ is continuous, it is straightforward to prove this function is well-defined. To prove that $\phi$ is a homomorphism, suppose any $[g], [h] \in \pi_1(X, x)$. Then,

$$\phi([g][h]) = \phi([g * h]) = [f \circ (g * h)]$$

By Lemma 4.1,

$$= [(f \circ g) * (f \circ h)] = [f \circ g][f \circ h] = \phi([g])\phi([h])$$

Now, define $\eta : \pi_1(Y, f(x)) \to \pi_1(X, x)$ by $\eta([g]) = [f^{-1} \circ g]$. Note that we are guaranteed the inverse function $f^{-1}$ since $f$ is a homeomorphism and is therefore bijective. In a similar manner as above, $\eta$ is shown to be a well-defined homomorphism. Furthermore, if we suppose any $[g] \in \pi_1(Y, f(x))$ we have

$$\phi(\eta([g])) = \phi([f^{-1} \circ g]) = [f \circ f^{-1} \circ g] = [g]$$

and if $[g] \in \pi_1(X, x)$, then

$$\eta(\phi([g])) = \eta([f \circ g]) = [f^{-1} \circ f \circ g] = [g]$$

Therefore, $\eta = \phi^{-1}$. Thus, $\phi$ is an isomorphism from $\pi_1(X, x)$ to $\pi_1(Y, f(x))$. □

While we have demonstrated some correspondence between the homeomorphism of topological spaces and the isomorphism of their respective fundamental groups, the converse of Theorem 4.8 is actually false. The simplest example of this is to consider a space $X$ such that there exist two open, non-empty, disjoint sets of $X$, $A, B$, such that $A$ and $B$ are each path-connected with no continuous path between them. If we let $a \in A$ we can see that $\pi_1(X, a) \cong \pi_1(A, a)$, but $A$ is clearly not homeomorphic to $X$.

Despite this shortcoming, this theorem remains of great interest to us as it provides a valuable topological tool for determining when two spaces are homeomorphic. In particular, the contrapositive of this theorem allows us to use our knowledge of group theory to prove if two topological spaces are not homeomorphic to each other—an oftentimes non-trivial task.

**Example 4.9:** Let $X$ be the set of all points in $\mathbb{R}^2$ of distance less than or equal to $\frac{3}{2}$ from the origin, and let $X$ have the relative topology as a subset of $\mathbb{R}^2$. Let $A$ be the set of all points in $\mathbb{R}^2$ of distance less than $\frac{1}{2}$ from the origin and let $Y = X \backslash A$ have the relative topology as a subset of $\mathbb{R}^2$. It should be intuitively clear that $X$ is not homeomorphic to $Y$. To prove this rigorously, however, we must use Theorem 4.8.

Considering $X$, we can show that $\pi_1(X) \cong \{e\}$ by using a deformable retraction. Let $r : X \to \{(0,0)\}$ be a constant map. Under the relative topology this is clearly continuous, since the inverse image under $r$ of any non-empty open set in $\{(0,0)\}$ is simply $X$, an open set. Then we can build a homotopy $H : X \times I \to X$ taking $X$ to $\{(0,0)\}$ by considering every point in $X$ as a vector and letting $H$ multiply that vector by $1 - t$ for all $t \in I$. Clearly, $\pi_1(\{(0,0)\}) \cong \{e\}$, and by Theorem 4.5, therefore $\pi_1(X) \cong \{e\}$.

Considering $Y$, we will use a deformable retraction to instead show that $\pi_1(Y) \cong \mathbb{Z}$. Clearly, $S^1 \subset Y$. If we let $r : Y \to S^1$ by considering the elements on $Y$ as vectors and letting $r$ set their distance to be 1 from the origin, it is clear that $r|_{S^1} = id_{S^1}$ and it is not difficult to prove that $r^{-1}(U)$ is open in $Y$ for all open sets $U \subseteq S^1$. Furthermore, we can create a homotopy from $Y$ to $S^1$ which acts like $r$, but instead of mapping the elements to a distance of 1 immediately, we slowly change their distance from the origin linearly with respect to $t$. By Theorem 4.5, it must be that $\pi_1(Y) \cong \pi_1(S^1) \cong \mathbb{Z}$. Note that it is impossible to define an equivalent $r$ or $H$ on $X$ since they are undefined for $(0,0)$, highlighting why these two spaces are indeed different.

By Theorem 4.8, since $\pi_1(X) \not\cong \pi_1(Y)$, $X$ and $Y$ are therefore not homeomorphic. ■

# 5 The Seifert-van Kampen Theorem

So far, we have seen several methods for calculating the fundamental group of a topological space. These methods only extend so far however, as they are generally only applicable to relatively simple spaces or spaces which can be formed as the product of simpler spaces. In more complicated spaces which cannot be easily represented, it is challenging to calculate the fundamental group of such a space using only the theorems we know. This brings us to the next theorem which, despite only being applicable in certain scenarios, provides us an immensely powerful tool for computing the fundamental group of topological spaces.

**Theorem 5.1: (Seifert-van Kampen Theorem)** Suppose $X$ is a path connected space such that there exist non-empty, path connected, open subsets $X_1, X_2$ of $X$ such that $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$ is nonempty and path connected. Let $x_0 \in X_0$, and let $\phi_1 : \pi_1(X_0, x_0) \hookrightarrow \pi_1(X_1, x_0)$ and $\phi_2 : (X_0, x_0) \hookrightarrow \pi_1(X_2, x_0)$ be the inclusion homomorphisms. If $\langle A|R_A\rangle$ and $\langle B|R_B\rangle$ are the presentations of $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$ respectively, then $\mathcal{G} = \langle A, B \mid R_A, R_A,\ \phi_1([\alpha])\phi_2([\alpha])^{-1} \text{ where } [\alpha] \in \pi_1(X_0, x_0)\rangle$ is a presentation for $\pi_1(X, x_0)$.

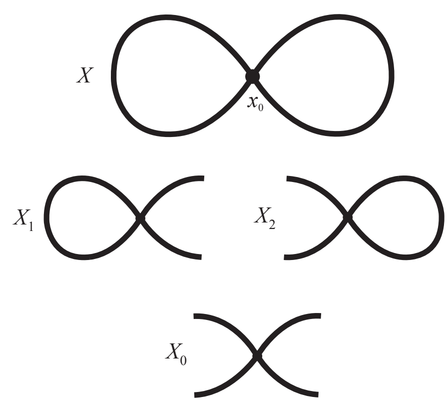
We will not directly prove this theorem here as it requires more advanced topological concepts than those discussed in this paper. Instead, we provide the commutative diagram below to illustrate this theorem. In this diagram, the function $\widetilde{f}$ is a naturally induced homomorphism from $\mathcal{G}$ to $\pi_1(X, x_0)$ and is the primary result of this theorem. The challenge of this theorem is proving $\widetilde{f}$ is a bijective function and therefore an isomorphism. For a full proof, we direct the reader to [1] or [2].

$$\begin{tikzcd}
 & & \pi_1(X, x_0) & & \\
 & & \mathcal{G} \arrow[u, dashed, "\widetilde{f}"'] & & \\
\pi_1(X_1, x_0) \arrow[uurr, bend left] \arrow[urr, dashed] & & & & \pi_1(X_2, x_0) \arrow[uull, bend right] \arrow[ull, dashed] \\
 & & \pi_1(X_0, x_0) \arrow[ull, "\phi_1"] \arrow[urr, "\phi_2"'] & &
\end{tikzcd}$$

Because of this theorem's somewhat strict requirements, it is not always possible to use it to calculate the fundamental group of a space. For example, we cannot use the Seifert-van Kampen Theorem to prove that $\pi_1(S^1) \cong \mathbb{Z}$ since there is no way to break $S^1$ into two open subsets $X_1$ and $X_2$ such that $X_1 \cup X_2 = S^1$ and $X_1 \cap X_2$ is path connected. Regardless, if a space meets the requirements of the theorem, the theorem proves to be an exceptionally powerful and succinct way to calculate the fundamental group, as we will see in this next example.

**Example 4.6:** Let $X \subset \mathbb{R}^2$ be the figure-eight shown in the diagram below with the relative topology and let $x_0 \in X$ be as indicated on the diagram. Furthermore, let $X_1$ and $X_2$ be the open sets as marked below. Note that $X$, $X_1$, $X_2$, and $X_0 = X_1 \cap X_2$ are each path connected and contain $x_0$. It should be clear that both $X_1$ and $X_2$ have a deformable retraction to a subset homeomorphic to $S^1$. Therefore, by Theorem 4.5 and Theorem 4.8, $\pi_1(X_1, x_0) \cong \pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(X_2, x_0) \cong \pi_1(S^1) \cong \mathbb{Z}$. Given this, let the presentations of $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$ be $\langle a | \rangle$ and $\langle b | \rangle$ respectively. Note that neither presentation has any relations apart from $e$.

Considering $X_0$, it is apparent that $X_0$ has a deformable retraction to the subset $\{x_0\}$. Therefore, $\pi_1(X_0, x_0) \cong \{e\}$. Then given $\phi_1$ and $\phi_2$ as defined in the Seifert-van Kampen Theorem, the set of relators $\{\phi_1([\alpha])\phi_2([\alpha])^{-1}$ where $[\alpha] \in \pi_1(X_0, x_0)\} = \{e\}$. Therefore, by the Seifert-van Kampen theorem, $\pi_1(X, x_0) \cong \langle a, b \rangle$. ■




# 6 Conclusion

In this paper, we have demonstrated how using only the most elementary definitions and concepts in point-set topology we can construct an entirely new methodology for studying and classifying topological spaces through the application of group theory. As we have seen, however, we cannot completely classify topological spaces up to homeomorphism through their fundamental groups alone. The fundamental group itself is only the first of what are called the homotopy groups in algebraic topology, which themselves are only one component of algebraic topology.

We hope this paper has provided the reader with an understanding of the fundamental group and why it is so important to algebraic topology, as well as enabled the reader's further exploration in point-set topology and algebraic topology.

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