# The Fundamental Group of Topological Spaces

## An Introduction To Algebraic Topology

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# An Introduction To Point-Set Topology

## Definition of Topological Spaces

### Definition:

A *topological space* is a nonempty set $X$ paired with a collection of subsets of $X$ called *open sets* satisfying:

- $X$ and $\varnothing$ are both open sets.
- The finite or infinite union of any collection of open sets is itself an open set.
- The finite intersection of any collection of open sets is itself an open set.

# An Introduction To Point-Set Topology

Examples of Topological Spaces

## Example:

We can define a topology on $\mathbb{R}^n$ to be the set of all possible arbitrary unions and finite intersections of open sets of the form:

$$U = \{x \in \mathbb{R}^n : d(x, y) < \varepsilon\}$$

for any $y \in \mathbb{R}^n, \varepsilon > 0$, where $d$ is the Euclidean distance function.

## Example:

We can define a topology on the set $I = [0, 1]$ as a subset of $\mathbb{R}^1$ by letting a set $U$ be open in $I$ if and only if there exists an open set $V$ in $\mathbb{R}^1$ such that $I \cap V = U$.

# An Introduction To Point-Set Topology

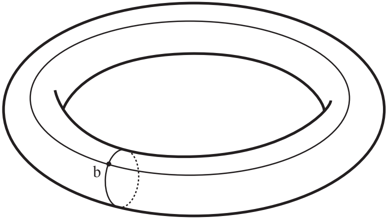
## Definition of Product Spaces

### Definition:

The *product space* of two topological spaces $X$ and $Y$ is the topological space $X \times Y$.
The topology on $X \times Y$ is the set of all possible arbitrary unions and finite intersections of open sets $U \times V$, where $U$ is an open set in $X$ and $V$ is an open set in $Y$.

### Example:

If we let $S^1 = \{x \in \mathbb{R}^2 : d(x, 0) = 1\}$ be the unit circle, then we can build the surface of a torus in $\mathbb{R}^3$ as the product topology $S^1 \times S^1$.



# An Introduction To Point-Set Topology

## Definition of Continuous Functions

### Definition:

If $X$ and $Y$ are topological spaces and $f : X \to Y$, then $f$ is a *continuous* function if $f^{-1}(V)$ is an open set in $X$ for all open sets $V$ in $Y$.

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- If we are working with functions to and from $\mathbb{R}^n$ or any subsets of $\mathbb{R}^n$, this definition of continuity is identical to the epsilon-delta definition.

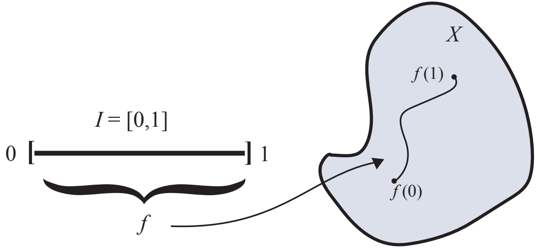
# Constructing The Fundamental Group

## Preliminary Definitions - Paths and Loops

At this point, we understand enough topology to describe the construction of the fundamental group of a topological space.

### Definition:

If $X$ is a topological space and $f : I \to X$ is a continuous function, then $f$ is a *path* in $X$. Given a path $f$, $f(0)$ and $f(1)$ are respectively the *initial point* and *terminal point* of the path. If $f$ is a path such that $f(0) = f(1)$, then $f$ is a *loop* based at the point $f(0)$.



# Constructing The Fundamental Group

Preliminary Definitions - Homotopic Functions

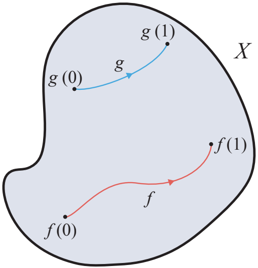
## Definition:

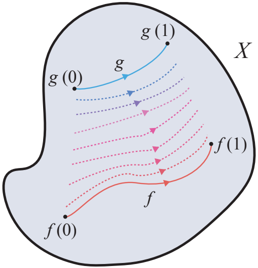
Given a topological space $X$ and paths $f, g : I \to X$, the functions $f$ and $g$ are *homotopic* to each other if there exists a continuous function $H : I \times I \to X$ such that:

- $H(i, 0) = f(i)$ for all $i \in I$
- $H(i, 1) = g(i)$ for all $i \in I$.

In this scenario, $H$ is the *homotopy* from $f$ to $g$.

If $f, g : I \to X$ are paths, then a homotopy $H : I \times I \to X$ from $f$ to $g$ can be thought of as a function which continuously deforms the function $f$ into the function $g$.



# Constructing The Fundamental Group

We can now construct the fundamental group:

- Let $X$ be a topological space and let $b \in X$ be any point in $X$.
- Let $L$ be the set of all loops $f : I \to X$ such that $f(0) = f(1) = b$.
- Define an operation $*$ on $L$ by $(f * g)(t) = \begin{cases} f(2t) & \text{for } t \in [0, \frac{1}{2}] \\ g(2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$
- Define a relation $\sim$ on $L$ where $f \sim g$ if and only if there exists a homotopy from $f$ to $g$.
  - This is an equivalence relation.
- Let $L/\sim$ be the set of equivalence classes of $L$ under $\sim$. This is the set component of the fundamental group.
- Define an operation on $L/\sim$ by: $[f][g] = [f * g]$, for all $[f], [g] \in L/\sim$.

## Definition:

The *fundamental group of the topological space X based at the point b* is the set $L/\sim$ combined with the operation of conjugation as defined above. This is denoted by $\pi_1(X, b)$.

# Constructing The Fundamental Group

To show that the fundamental group is actually a group:

- To show the operation defined on $\pi_1(X,b)$ is associative, we must show that $[f]\left([g][h]\right) = \left([f][g]\right)[h]$. This is the same as showing that $f * (g * h) \sim (f * g) * h$.

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- If $c : I \to X$ is the constant map defined by $c(i) = b$, then $[c]$ is the identity element of $\pi_1(X, b)$. To show this, we show that $[f][c] = [f]$, which is the same as showing that $f * c \sim f$.

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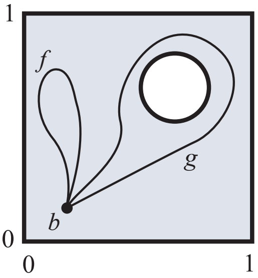
- If $[f] \in \pi_1(X, b)$, if we let $f^r : I \to X$ by $f^r(i) = f(1 - i)$, then $[f]^{-1} = [f^r]$. To show this, we show that $[f][f^r] = [c]$. That is, $f * f^r \sim c$.

# Examples

## Fundamental Group Of A Square With A Hole

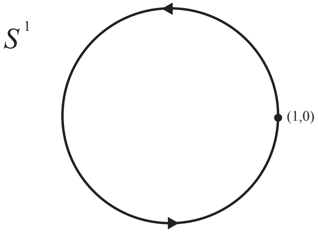
### Example:

- Consider the space $I \times I$ with a disc removed, as shown below.
- If we make a loop $f$ without looping around the hole, we can continuously deform $f$ back into $c$.
- However, if we loop $g$ around the hole it is impossible to continuously deform $g$ back into $c$.
- Therefore, $[f] \neq [g]$.



# Examples

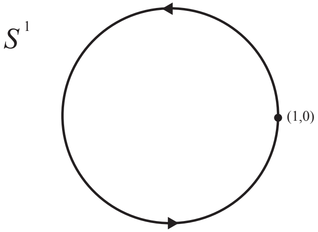
## Fundamental Group Of $S^1$




- Consider the space $S^1$.

# Examples

## Fundamental Group Of $S^1$




- Consider the space $S^1$.
- If we make one full rotation around $S^1$, it is impossible to continuously deform that loop back into the constant map (identity element).

# Examples

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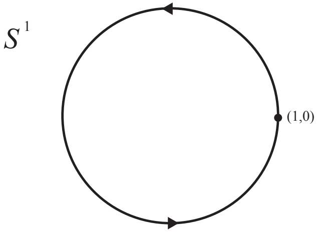


- Consider the space $S^1$.
- If we make one full rotation around $S^1$, it is impossible to continuously deform that loop back into the constant map (identity element).
- If we make two full rotations in the same direction, we cannot deform the rotations back into one rotation or the identity.

# Examples

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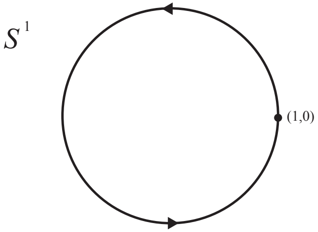


- Consider the space $S^1$.
- If we make one full rotation around $S^1$, it is impossible to continuously deform that loop back into the constant map (identity element).
- If we make two full rotations in the same direction, we cannot deform the rotations back into one rotation or the identity.
- If we make one rotation in the opposite direction, we undo the last rotation.

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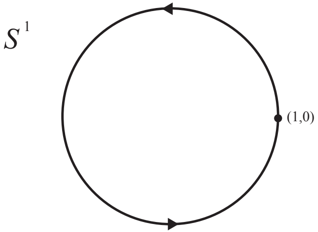



- Consider the space $S^1$.
- If we make one full rotation around $S^1$, it is impossible to continuously deform that loop back into the constant map (identity element).
- If we make two full rotations in the same direction, we cannot deform the rotations back into one rotation or the identity.
- If we make one rotation in the opposite direction, we undo the last rotation.
- From this, we assert that $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$.

# Examples

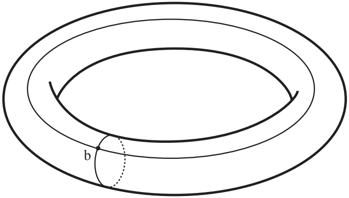
## The Product Topology And Group Products

### Theorem:

*If $X$ and $Y$ are topological spaces and $x \in X$ and $y \in Y$, then* $\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y)$.

### Example:

Since we can represent the surface of a torus topologically as $S^1 \times S^1$, therefore the fundamental group of a torus is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.



# The Seifert-van Kampen Theorem

## Definition:

A topological space $X$ is *path connected* if for all $x, y \in X$, there exists a path $f : I \to X$ such that $f(0) = x$ and $f(1) = y$.

## Theorem:

- Let $X$ be a path connected topological space such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 \neq \varnothing$ for path connected open sets $X_1, X_2$.
- Let $X_0 = X_1 \cap X_2$ and let $x_0 \in X_0$. We require that $X_0$ is path connected.
- Let
$$\phi_1 : \pi_1(X_0, x_0) \hookrightarrow \pi_1(X_1, x_0) \ \text{ by } \ \phi_1([f]) = [f]$$
and let
$$\phi_2 : \pi_1(X_0, x_0) \hookrightarrow \pi_1(X_2, x_0) \ \text{ by } \ \phi_2([f]) = [f]$$
be the inclusion homomorphisms.
- Let $\langle A | R_A \rangle$ and $\langle B | R_B \rangle$ be presentations for $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$ respectively.
- Then, $\mathcal{G}$ is a presentation for $\pi_1(X, x_0)$, where

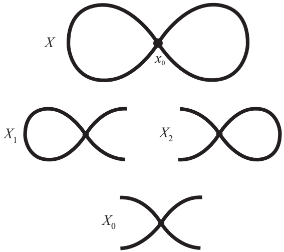
$$\mathcal{G} = \langle A, B \mid R_A, R_B, \ \phi_1([\alpha])\phi_2([\alpha])^{-1} \text{ where } [\alpha] \in \pi_1(X_0, x_0) \rangle$$

# Example

## The Fundamental Group Of A Figure Eight

### Example:

- Let $X$ be the figure eight shown below and let $X_1, X_2$, and $X_0$ be the open subsets of $X$ shown below.
- $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$ are both isomorphic to $\mathbb{Z}$.
  - So, $\langle a | \rangle$ and $\langle b | \rangle$ are the presentations for $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$.
- $\pi_1(X_0, x_0) \cong \{e\}$, implying that the only relations of the form $\phi_1([\alpha])\phi_2([\alpha])^{-1}$ where $[\alpha] \in \pi_1(X_0, x_0)$ are $[c]$, the identity element.
- Therefore, $\langle a, b | \rangle$ is the presentation for $\pi_1(X, x_0)$.



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