Math 290 Name: Key Dr. Beezer
Exam 6 Spring 2015
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use Sage to manipulate and row-reduce matrices. Be sure to make it clear what you have input to Sage, and show any output you use to justify your answers.

1. Consider the following questions about the linear transformation T. (30 points)

$$T\colon \mathbb{C}^2 \to \mathbb{C}^3, \qquad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a+b \\ -2a-b \\ 4a+2b \end{bmatrix}$$

(a) Compute the kernel of T, $\mathcal{K}(T)$.

$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ system $\begin{matrix} 2a+b=0 \\ -2a-b=0 \\ 4a+2b=0 \end{matrix}$ RREF coeff matrix $\rightarrow \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\mathcal{K}(T) = \left\langle \left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\} \right\rangle$

(b) Based on your answer to the previous question, is T injective?

Theorem KILT, $\mathcal{K}(T) \neq \{0\} \Rightarrow T$ is <u>not</u> injective

(c) Find two vectors $\mathbf{x}$ and $\mathbf{y}$ such that $T(\mathbf{x}) = T(\mathbf{y})$.

$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Theorem LTTZZ $\qquad$ in $\mathcal{K}(T)$

(d) Compute the dimension of the range of T, $\dim(\mathcal{R}(T))$.

$\dim(\mathcal{K}(T)) = 1$

$2 = \dim(\mathbb{C}^2) = \dim(\mathcal{R}(T)) + \dim(\mathcal{K}(T)) = \dim \mathcal{R}(T) + 1$

domain $\quad$ Theorem RPNDD $\qquad \Rightarrow \dim \mathcal{R}(T) = 1$

(e) Based on your answer to the previous question, is T surjective?

$\dim \mathcal{R}(T) = 1 \neq 3 = \dim(\mathbb{C}^3) \Rightarrow \mathcal{R}(T) \neq \mathbb{C}^3$

codomain

so by Theorem RSLT, T is <u>not</u> surjective.

(f) Find a vector $\mathbf{x}$ whose preimage, $T^{-1}(\mathbf{x})$, is empty.

<u>Every</u> output is a multiple of $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, so for example,

$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ leads to a system of three equations in a & b with no solution, hence $T^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$ is empty.

2. Given $R\colon \mathbb{C}^3 \to P_2$ below, find an explicit formula for values of R^{-1}. You may assume that R is invertible. (P_2 is the vector space of polynomials with degree at most 2.) (20 points)

$$R\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (3a + b - 6c) + (2a + b - 5c)\,x + (-3a + 2b - 2c)\,x^2$$

Basis of $P_2 = \{1, x, x^2\}$

Preimages

$$T^{-1}(1) = \left\{\begin{bmatrix} 8 \\ 19 \\ 7 \end{bmatrix}\right\} \qquad T^{-1}(x) = \left\{\begin{bmatrix} -10 \\ -24 \\ -9 \end{bmatrix}\right\} \qquad T^{-1}(x^2) = \left\{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right\}$$

$$T^{-1}(a + bx + cx^2) = a\,T^{-1}(1) + b\,T^{-1}(x) + c\,T^{-1}(x^2)$$

$$= a\begin{bmatrix} 8 \\ 19 \\ 7 \end{bmatrix} + b\begin{bmatrix} -10 \\ -24 \\ -9 \end{bmatrix} + c\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8a - 10b + c \\ 19a - 24b + 3c \\ 7a - 9b + c \end{bmatrix}$$

Each of these comes from solving a linear system with a nonsingular coefficient matrix.

3. Verify that the function below is a linear transformation. (20 points)

$$S\colon \mathbb{C}^3 \to \mathbb{C}^2,\ S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a - 2b \\ b + 4c \end{bmatrix}$$

Two conditions to check:

① $$T(\underset{\sim}{x} + \underset{\sim}{y}) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (x_1 + y_1) - 2(x_2 + y_2) \\ x_2 + y_2 + 4(x_3 + y_3) \end{bmatrix} = \begin{bmatrix} (x_1 - 2x_2) + (y_1 - 2y_2) \\ (x_2 + 4x_3) + (y_2 + 4y_3) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - 2x_2 \\ x_2 + 4x_3 \end{bmatrix} + \begin{bmatrix} y_1 - 2y_2 \\ y_2 + 4y_3 \end{bmatrix} = T(\underset{\sim}{x}) + T(\underset{\sim}{y})$$ for all $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{C}^3$

② $$T(\alpha \underset{\sim}{x}) = T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \alpha x_1 - 2(\alpha x_2) \\ \alpha x_2 + 4(\alpha x_3) \end{bmatrix} = \begin{bmatrix} \alpha(x_1 - 2x_2) \\ \alpha(x_2 + 4x_3) \end{bmatrix} = \alpha \begin{bmatrix} x_1 - 2x_2 \\ x_2 + 4x_3 \end{bmatrix} = \alpha T(\underset{\sim}{x})$$

for all $\alpha \in \mathbb{C}$, $\underset{\sim}{x} \in \mathbb{C}^3$.

4. Suppose that $T\colon U \to W$ is a linear transformation such that $T(\alpha\mathbf{u}) = \alpha^2 T(\mathbf{u})$ for all $\alpha \in \mathbb{C}$ and all $\mathbf{u} \in U$. Prove that $T(\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u} \in U$. (15 points)

$$\alpha^2 T(\underset{\sim}{u}) \underset{\text{given}}{=} T(\alpha \underset{\sim}{u}) \underset{T \text{ is L.T.}}{=} \alpha T(u)$$

$$\Rightarrow \alpha^2 T(\underset{\sim}{u}) - \alpha T(\underset{\sim}{u}) = \underset{\sim}{0}$$

$$\Rightarrow \alpha(\alpha - 1) T(\underset{\sim}{u}) = \underset{\sim}{0} \quad \text{for all } \alpha, \text{ all } \underset{\sim}{u}$$

Choose some $\alpha \neq 0, 1$. Then by theorem SMEZV,

$$T(\underset{\sim}{u}) = \underset{\sim}{0} \quad \text{for all } \underset{\sim}{u} \in U$$

5. Suppose that $T\colon V \to W$ is an injective linear transformation and that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ is a linearly independent subset of V. Prove that $R = \{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), \ldots, T(\mathbf{v}_m)\}$ is a linearly independent subset of W. (15 points)

RLD: $a_1 T(\underset{\sim}{v_1}) + a_2 T(\underset{\sim}{v_2}) + \cdots + a_m T(\underset{\sim}{v_m}) = \underset{\sim}{0}$

$$T(a_1 \underset{\sim}{v_1} + a_2 \underset{\sim}{v_2} + \cdots + a_m \underset{\sim}{v_m}) = \underset{\sim}{0} \quad \text{Theorem LTLC}$$

$$\Rightarrow a_1 \underset{\sim}{v_1} + a_2 \underset{\sim}{v_2} + \cdots + a_m \underset{\sim}{v_m} = \underset{\sim}{0} \quad \text{Theorem KILT}$$

a RLD on linearly independent $\underset{\sim}{S}$, so

$$a_1 = a_2 = \cdots = a_m = 0$$

thus, T is linearly independent