1. Consider the following questions about the linear transformation $T$. (30 points)

$$T: \mathbb{C}^2 \to \mathbb{C}^3, \quad T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} 2a + b \\ -2a - b \\ 4a + 2b \end{bmatrix}$$

(a) Compute the kernel of $T$, $\mathcal{K}(T)$.

(b) Based on your answer to the previous question, is $T$ injective?

(c) Find two vectors $x$ and $y$ such that $T(x) = T(y)$.

(d) Compute the dimension of the range of $T$, $\dim(\mathcal{R}(T))$.

(e) Based on your answer to the previous question, is $T$ surjective?

(f) Find a vector $x$ whose preimage, $T^{-1}(x)$, is empty.
2. Given \( R: \mathbb{C}^3 \to P_2 \) below, find an explicit formula for values of \( R^{-1} \). You may assume that \( R \) is invertible. (\( P_2 \) is the vector space of polynomials with degree at most 2.) (20 points)

\[
R \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (3a + b - 6c) + (2a + b - 5c)x + (-3a + 2b - 2c)x^2
\]

3. Verify that the function below is a linear transformation. (20 points)

\( S: \mathbb{C}^3 \to \mathbb{C}^2 \), \( S \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a - 2b \\ b + 4c \end{bmatrix} \)
4. Suppose that \( T: U \rightarrow W \) is a linear transformation such that \( T(\alpha u) = \alpha^2 T(u) \) for all \( \alpha \in \mathbb{C} \) and all \( u \in U \). Prove that \( T(u) = 0 \) for all \( u \in U \). (15 points)

5. Suppose that \( T: V \rightarrow W \) is an injective linear transformation and that \( S = \{ v_1, v_2, v_3, \ldots, v_m \} \) is a linearly independent subset of \( V \). Prove that \( R = \{ T(v_1), T(v_2), T(v_3), \ldots, T(v_m) \} \) is a linearly independent subset of \( W \). (15 points)