Math 290 | Name: Key | Dr. Beezer
Exam 5 | | Spring 2015
Chapter D&E

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Instructions on Sage usage accompanies each problem. Be sure to make it clear what you have input to Sage, and show any output you use to justify your answers.

1. Compute the determinant of the matrix below "by hand" (i.e. without using Sage to justify any part of your answer). (10 points)

$$A = \begin{bmatrix} 2 & 2 & -6 \\ 4 & 7 & -8 \\ 2 & 3 & -5 \end{bmatrix}$$

About row 1

$$\det(A) = 2(1)\begin{vmatrix} 7 & -8 \\ 3 & -5 \end{vmatrix} + 2(-1)\begin{vmatrix} 4 & -8 \\ 2 & -5 \end{vmatrix} + -6(1)\begin{vmatrix} 4 & 7 \\ 2 & 3 \end{vmatrix}$$

$$= 2(-11) - 2(-4) - 6(-2) = -22 + 8 + 12 = -2$$

2. Determine if the matrices below are diagonalizable or not. Your answer **must** contain justification **based on theorems** from this part of the course (or earlier). You may use Sage to provide most any **computation** you need. (So, for example, an answer that simply cites Sage output from `.is_diagonalizable()` would get zero credit, while you should feel free to compute characteristic polynomials, eigenvalues, and eigenspaces so long as you describe the output carefully and interpret it properly in the context of theorems.) (20 points)

(a) $B = \begin{bmatrix} 32 & -40 & 55 & -75 \\ 50 & -78 & 25 & -125 \\ -10 & 15 & -8 & 25 \\ -20 & 35 & 5 & 52 \end{bmatrix}$

B.fcp() $\rightarrow (x-2)^2(x+3)^2$

B.eigenspaces_right() : $\gamma_B(2) = 2$, $\gamma_B(-3) = 2$

algebraic & geometric multiplicities are <u>equal</u> for <u>each</u> eigenspace. Theorem DMFE $\Rightarrow$ B is diagonalizable

You <u>could</u> use .eigenmatrix_right() to get a linearly independent set of 5 eigenvectors, but it would require the right checks.

(b) $C = \begin{bmatrix} -27 & 71 & -9 & 99 & -29 \\ -31 & 39 & -3 & 33 & 65 \\ -31 & 31 & -4 & 12 & 89 \\ 11 & -1 & -1 & 13 & -53 \\ -2 & 13 & -2 & 22 & -20 \end{bmatrix}$

C.fcp() $\rightarrow (x-2)^2(x+1)^3 \Rightarrow \alpha_C(-1) = 3$

C.eigenspaces_right() : $\gamma_C(-1) = 1$

so $\alpha_C(-1) \neq \gamma_C(-1)$

By Theorem DMFE, C is not diagonalizable.

Theorem DC, Definition DM are not helpful here.

3. Consider the 6×6 matrix E below. (30 points)

$$E = \begin{bmatrix} -16 & -36 & -1 & 33 & 9 & -60 \\ 7 & 14 & -1 & -16 & -6 & 24 \\ -9 & 0 & 8 & 36 & 27 & -36 \\ 6 & 0 & -7 & -25 & -18 & 21 \\ -8 & -12 & 4 & 22 & 11 & -25 \\ 3 & 0 & -2 & -12 & -9 & 14 \end{bmatrix}$$

(a) Use Sage to produce a factored version of the characteristic polynomial. From this information alone, list the eigenvalues of E and their algebraic multiplicities, explaining clearly how they are being derived from the output you have listed from Sage.

$$E.\text{fcp}() \rightarrow (x+1)^2 (x-2)^4 = (x-(-1))^2 (x-2)^4$$

$$\alpha_E(-1) = 2 \qquad \alpha_E(2) = 4$$

(b) Choose one of the positive eigenvalues of E and compute a basis for the eigenspace of the eigenvalue. Limit your justifications from Sage to the `.rref()` method.

$$E - 2I_6 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & -2 \\ 0 & 1 & 0 & 0 & 5/4 & 3/4 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_E(2) = \left\langle \left\{ \begin{bmatrix} \\ \\ \\ \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} 3 \\ -5/4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3/4 \\ -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

(c) Compute the geometric multiplicity of each eigenvalue of E. Limit your justifications from Sage to the `.rref()` method.

$$E - (-1)I_6 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 5

nullity = 1

$$\gamma_E(-1) = \dim(N(E-(-1)I_6)) = 1$$

From basis above, $\gamma_E(2) = 2$

4. Suppose that A is a matrix, λ is an eigenvalue of A, and $\alpha \in \mathbb{C}$ is a scalar. Prove that $\lambda + \alpha$ is an eigenvalue of $A + \alpha I_n$. (15 points)

Suppose $\underset{\sim}{x}$ is an eigen vector of A for λ, then

$$
\begin{aligned}
(A+\alpha I_n)\underset{\sim}{x} &= Ax + \alpha I \underset{\sim}{x} \\
&= \lambda \underset{\sim}{x} + \alpha \underset{\sim}{x} \\
&= (\lambda+\alpha)\underset{\sim}{x}
\end{aligned}
$$

demonstrating that $\lambda + \alpha$ is an eigenvalue of $A+\alpha I_n$

5. Suppose that A and B are similar matrices, and $\alpha \in \mathbb{C}$ is a scalar. Prove that $A + \alpha I_n$ and $B + \alpha I_n$ are similar matrices. (15 points)

By hypothesis, there exist S so that $S^{-1}AS = B$. Then

$$
\begin{aligned}
S^{-1}(A+\alpha I_n)S &= S^{-1}AS + S^{-1}\alpha I_n S \\
&= S^{-1}AS + \alpha S^{-1}S \\
&= B + \alpha I_n,
\end{aligned}
$$

as desired.

6. Suppose that A is a square matrix with characteristic polynomial $p_A(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$. Prove that $\det(A) = a_0$. (10 points)

$$
\begin{aligned}
a_0 &= a_n(0)^n + a_{n-1}(0)^{n-1} + \cdots + a_1 \cdot 0 + a_0 \\
&= p_A(0) \\
&= \det(A - 0I) \\
&= \det(A).
\end{aligned}
$$