Math 290 Name: Key Dr. Beezer
Exam 2 Spring 2015
Chapter V

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use Sage to manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Is the set R linearly independent or not? (15 points)

$$R = \left\{ \begin{bmatrix} -4 \\ -5 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ -1 \\ -4 \end{bmatrix} \right\}$$

Theorem LIVRN suggest making a matrix w/ these vectors as columns & row-reducing this matrix.

$$[\underset{\sim}{v_1} | \underset{\sim}{v_2} | \underset{\sim}{v_3}] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$r = 2 \neq 3 = n$

so the set is linearly dependent

2. Is the vector $\mathbf{u}$ in the span of P, $\langle P \rangle$? (15 points)

$$\mathbf{u} = \begin{bmatrix} -11 \\ -6 \\ 5 \\ 10 \end{bmatrix} \qquad P = \left\{ \begin{bmatrix} -3 \\ -2 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 8 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\} = \{\underset{\sim}{u_1}, \underset{\sim}{u_2}, \underset{\sim}{u_3}\}$$

Find $\alpha_1, \alpha_2, \alpha_3$ so that

Defn SSCV $\longrightarrow \alpha_1 \underset{\sim}{u_1} + \alpha_2 \underset{\sim}{u_2} + \alpha_3 \underset{\sim}{u_3} = \underset{\sim}{u}$

By theorem SLSLC, solve system w/ augmented matrix,

$$\left[\begin{array}{ccc|c} -3 & 5 & 0 & -11 \\ -2 & 3 & 1 & -6 \\ -5 & 8 & 2 & 5 \\ -2 & -4 & 2 & 10 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 107 \\ 0 & 1 & 0 & 62 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So we see that the system has a unique solution.

So there <u>are</u> scalars $\alpha_1, \alpha_2, \alpha_3$, thus $\underset{\sim}{u} \in \langle P \rangle$.

(specifically $\underset{\sim}{u} = 107 \underset{\sim}{u_1} + 62 \underset{\sim}{u_2} + 22 \underset{\sim}{u_3}$

3. Find a linearly independent set T whose span is the same as the span of S, that is, $\langle T\rangle = \langle S\rangle$. (20 points)

$$S = \left\{ \begin{bmatrix}1\\-1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\3\\-2\end{bmatrix}, \begin{bmatrix}1\\-3\\4\\4\end{bmatrix}, \begin{bmatrix}-2\\0\\-1\\3\end{bmatrix}, \begin{bmatrix}4\\-1\\3\\-4\end{bmatrix}, \begin{bmatrix}5\\-1\\4\\-6\end{bmatrix}, \begin{bmatrix}0\\3\\-5\\-4\end{bmatrix}, \begin{bmatrix}-2\\1\\-3\\2\end{bmatrix} \right\}$$

Theorem BS suggest row-reducing a matrix with the vectors of S as the columns.

$$[\mathbf{v}_1|\mathbf{v}_2|\dots|\mathbf{v}_8] \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 5 & 0 & 2 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

, then $D = \{1, 2, 4\}$ (pivot column locations)

Set $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\} = \left\{ \begin{bmatrix}1\\-1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\3\\-2\end{bmatrix}, \begin{bmatrix}-2\\0\\-1\\3\end{bmatrix} \right\}$.

Theorem BS guarantees that ① T is linearly independent
② $\langle T\rangle = \langle S\rangle$.

4. Find a linearly independent set of vectors T that span the null space of A, that is, $\langle T\rangle = \mathcal{N}(A)$. (20 points)

$$A = \begin{bmatrix} -1 & 0 & -4 & 2 & 6 & -2 & 0 \\ 1 & -1 & -1 & -3 & -5 & 2 & -1 \\ 0 & -1 & -5 & 0 & 6 & 0 & 1 \\ 1 & -2 & -6 & -3 & 1 & 3 & 1 \end{bmatrix}$$

Solve the homogenous system $\mathcal{LS}(A, \mathbf{0})$ by row-reducing the coefficient matrix.

$$A \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 4 & 0 & 4 & 0 & 2 \\ 0 & 1 & 5 & 0 & -6 & 0 & -1 \\ 0 & 0 & 0 & 1 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$D = \{1, 2, 4, 6\}$, $F = \{3, 5, 7\}$

Apply Theorem BNS,

$$T = \left\{ \begin{bmatrix} \\ \\ 1 \\ \\ 0 \\ \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ \\ 1 \\ \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ \\ 0 \\ \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix}-4\\-5\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-4\\6\\0\\-5\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\-2\\0\\-1\\1\end{bmatrix} \right\}$$

Theorem BNS says
① T lin. ind.
② $\langle T\rangle = N(A)$

"pattern of zeros & ones" due to $F = \{3, 5, 7\}$

fill w/ negatives of non-pivot columns.

5. Construct a careful proof, using Definition CVE (Column Vector Equality), of the following: if $\mathbf{v} \in \mathbb{C}^m$, then $0\mathbf{v} = \mathbf{0}$. Be sure to provide justification for each step of your proof. (15 points)

For $1 \leq i \leq m$,

$$[0\underset{\sim}{v}]_i = 0[\underset{\sim}{v}]_i \qquad \text{Defn CVSM}$$
$$= 0 \qquad \text{Theorem ZPCN}$$
$$= [\underset{\sim}{0}]_i \qquad \text{Defn ZCV}$$

So by Defn CVE, $0\underset{\sim}{v} = \underset{\sim}{0}$.

6. Prove that the inner product is anti-commutative. That is, for $\mathbf{u}, \mathbf{w} \in \mathbb{C}^n$, $\langle \mathbf{u}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{u} \rangle}$. (15 points)

$$\overline{\langle w, u \rangle} = \overline{\sum_{i=1}^{n} [\underset{\sim}{w}]_i \overline{[\underset{\sim}{u}]_i}}$$
$$= \sum_{i=1}^{n} \overline{[\underset{\sim}{w}]_i \overline{[\underset{\sim}{u}]_i}}$$
$$= \sum_{i=1}^{n} \overline{[\underset{\sim}{w}]_i}\, \overline{\overline{[\underset{\sim}{u}]_i}}$$
$$= \sum_{i=1}^{n} \overline{[\underset{\sim}{w}]_i}\, [\underset{\sim}{u}]_i$$
$$= \sum_{i=1}^{n} [\underset{\sim}{u}]_i \overline{[\underset{\sim}{w}]_i}$$
$$= \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle$$

Can you supply an acronym/reason for each step?