Overview

1 Basics
   - Multilinearity
   - Dual Space

2 Tensors
   - Tensor Product
   - Basis of $\mathcal{T}_q^p(V)$

3 Component Representation
   - Kronecker Product
   - Components
   - Comparison
Multilinear Functions

Definition
A function $f : V \mapsto W$, where $V$ and $W$ are vector spaces over a field $F$, is linear if for all $x, y$ in $V$ and all $\alpha, \beta$ in $F$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$
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f(\alpha x_1 + \beta x_2, y) = \alpha f(x_1, y) + \beta f(x_2, y), \quad \text{and} \quad
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A function $f : V_1 \times \cdots \times V_s \mapsto W$, where \( \{ V_i \}_{i=1}^s \) and \( W \) are vector spaces over a field \( F \), is \( s \)-linear if for all \( x_i, y_i \) in \( V_i \) and all \( \alpha, \beta \) in \( F \)

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f(v_1, \ldots, \alpha x_i + \beta y_i, \ldots, v_s) = \\
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Example

We already know of a bilinear function from $V \times V \rightarrow \mathbb{R}$. Any inner product defined on $V$ is such a function, as $\langle \alpha v_1 + \beta v_2, u \rangle = \alpha \langle v_1, u \rangle + \beta \langle v_2, u \rangle$, and $\langle v, \alpha u_1 + \beta u_2 \rangle = \alpha \langle v, u_1 \rangle + \beta \langle v, u_2 \rangle$. 
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When treated as a function of the columns (or rows) of an $n \times n$ matrix, the determinant is $n$-linear.
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Example

For any collection of vector spaces $\{V_i\}_{i=1}^s$, and any collection of linear functions $f_i : V_i \rightarrow \mathbb{R}$, the function

$$f(v_1, \ldots, v_s) = \prod_{i=1}^{s} (f_i(v_i))$$

is $s$-linear.
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Definition $L(V; \mathbb{R})$ is the dual space of $V$, and is denoted $V^\ast$. 

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Dual Space

- $V^* \cong V$, as they are both vector spaces of dimension $N$. 

- Notation: $\langle v^*, v \rangle$ denotes the value of $v^*$ evaluated at $v$. For our purposes, consider it the inner product of $v$ and $v^*$.
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Davis Shurbert (UPS)
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Tensors

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A tensor of order \((p, q)\) is a \((p + q)\)-linear map

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T : V^* \times \cdots \times V^* \times V \times \cdots \times V \mapsto \mathbb{R}.
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- \(\mathcal{T}^0_1(V) = V^*\), \(\mathcal{T}^1_0(V) = V\), and \(\mathcal{T}^1_1(V) \cong L(V : V)\).
- That is, lower order tensors are the 1 and 2 dimensional arrays we usually work with.
Given any two vectors $v^* \in V^*$ and $v \in V$, we can construct a tensor of order $(1, 1)$. 

Consider the function $(v \otimes v^*) : V^* \times V \mapsto \mathbb{R}$, defined as 

$$(v \otimes v^*)(u^*, u) = \langle u^*, v \rangle \langle v^*, u \rangle$$

Recall that this is a special case of our earlier example, as $(v \otimes v^*)$ is the product of two linear functions.
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For any collection of vectors $\{v_i\}_{i=1}^p$, and vectors $\{v^j\}_{j=1}^p$, their tensor product is the function

$$v_1 \otimes \cdots \otimes v_p \otimes v^1 \otimes \cdots \otimes v^q : V^* \times \cdots \times V^* \times V \times \cdots \times V \mapsto \mathbb{R},$$

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$$(v_1 \otimes \cdots \otimes v_p \otimes v^1 \otimes \cdots \otimes v^q)(u^1, \ldots u^p, u_1, \ldots u_q)$$

$$= \langle u^1, v_1 \rangle \cdots \langle u^p, v_p \rangle \langle v^1, u_1 \rangle \cdots \langle v^q, u_q \rangle$$
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• However, we can use simple tensors to build a basis of $\mathcal{T}_d^p(V)$. 

Theorem

For any basis of $V$, $B = \{e_i\}_{i=1}^N$, there exists a unique dual basis of $V^*$ relative to $B$, denoted $\{e^j\}_{j=1}^N$ and defined as

$$\langle e^j, e_i \rangle = \delta_{j,i} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$
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for all combinations of \( \{ i_k \}_{k=1}^p \in \{ 1, \ldots, N \} \) and \( \{ j_z \}_{z=1}^q \in \{ 1, \ldots, N \} \), forms a basis of \( \mathcal{T}_q^p(V) \).
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for all combinations of \( \{ i_k \}_{k=1}^p \in \{ 1, \ldots, N \} \) and \( \{ j_z \}_{z=1}^q \in \{ 1, \ldots, N \} \), forms a basis of \( \mathcal{T}_{q}^{p}(V) \).

- The size of this basis is \( N^{(p+q)} \).
Theorem

For any basis \( \{ e_i \}_{i=1}^{N} \) of \( V \), and the corresponding dual basis \( \{ e^j \}_{j=1}^{N} \) of \( V^* \), the set of simple tensors

\[
\{ e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e_{j_1} \otimes \cdots \otimes e_{j_q} \}
\]

for all combinations of \( \{ i_k \}_{k=1}^{p} \in \{1, \ldots, N\} \) and \( \{ j_z \}_{z=1}^{q} \in \{1, \ldots, N\} \), forms a basis of \( T_q^p(V) \).

- The size of this basis is \( N^{(p+q)} \).
- Simplified proof in my paper, but our relation of linear dependence is nasty (\( p + q \) nested sums).
Kronecker Product

Can we use this basis to find a component representation of tensors in $\mathcal{T}_q^p(V)$?
Kronecker Product

Can we use this basis to find a component representation of tensors in $T_q^p(V)$? Yes, but first...

Definition

For two matrices $A_{m \times n}$ and $B_{p \times q}$, the Kronecker product of $A$ and $B$ is defined as

$$A \otimes B = \begin{pmatrix}
[A]_{1,1}B & [A]_{1,2}B & \cdots & [A]_{1,n}B \\
[A]_{2,1}B & [A]_{2,2}B & \cdots & [A]_{2,n}B \\
\vdots & \vdots & \ddots & \vdots \\
[A]_{m,1}B & [A]_{m,2}B & \cdots & [A]_{m,n}B
\end{pmatrix}$$

Can be represented by 2-dimensional array, but we consider this product to be a list of lists, table of lists, list of tables, table of tables, etc.
Kronecker Product

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- Can be represented by 2-dimensional array, but we consider this product to be a list of lists, table of lists, list of tables, table of tables, etc.
Components as Basis Images

Definition

In general, we define the components of $T \in T_q^p(V)$ to be the $(p + q)$-indexed scalars

$$A_{i_1, \ldots, i_p}^{j_1, \ldots, j_q} = A(e_1^i, \ldots, e_p^i, e_1^j, \ldots, e_q^j).$$
Components as Basis Images

Definition

In general, we define the components of \( T \in T_{q}^{p}(V) \) to be the \((p + q)\)-indexed scalars

\[
A_{i_{1}, \ldots, i_{p}}^{j_{1}, \ldots, j_{q}} = A(e^{i_{1}}, \ldots, e^{i_{p}}, e_{j_{1}}, \ldots, e_{j_{q}}).
\]

- For vectors, this is exactly how we define components \( \langle v, e_{i} \rangle = [v]_{i} \).
Definition

In general, we define the components of $T \in T_q^p(V)$ to be the $(p + q)$-indexed scalars

$$A_{i_1, \ldots, i_p}^{j_1, \ldots, j_q} = A(e_1^{i_1}, \ldots, e_p^{i_p}, e_1^{j_1}, \ldots, e_q^{j_q}).$$

- For vectors, this is exactly how we define components ($\langle v, e_i \rangle = [v]_i$).
- If $T$ is a simple tensor, then the $(p + q)$-dimensional array formed by $A_{i_1, \ldots, i_p}^{j_1, \ldots, j_q}$ is equal to the Kronecker product of the vectors and covectors which make up $T$. 
Example

For $V = \mathbb{R}^2$, consider the vectors $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v^* = \begin{bmatrix} 2 & 1 \end{bmatrix}$, and $w^* = \begin{bmatrix} 1 & 3 \end{bmatrix}$. Let $A = u \otimes v^* \otimes w^*$ and consider
Example

For $V = \mathbb{R}^2$, consider the vectors $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v^* = \begin{bmatrix} 2 & 1 \end{bmatrix}$, and $w^* = \begin{bmatrix} 1 & 3 \end{bmatrix}$. Let $A = u \otimes v^* \otimes w^*$ and consider

$$A_{1,1}^1 = A(e^1, e_1, e_1)$$
$$= \langle \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rangle \langle \begin{bmatrix} 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \langle \begin{bmatrix} 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$$
$$= 2$$

$$A_{1,2}^2 = A(e^1, e_2, e_1) = 1$$

$$A_{1,1}^2 = A(e^1, e_1, e_2) = 6$$

...
Example

Or, we can take the Kronecker product $u \otimes v^* \otimes w^*$ to get
Example

Or, we can take the Kronecker product $u \otimes v^* \otimes w^*$ to get

$$u \otimes v^* \otimes w^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \begin{bmatrix} 1 & 3 \end{bmatrix} & 1 \begin{bmatrix} 1 & 3 \end{bmatrix} \\ 2 \begin{bmatrix} 1 & 3 \end{bmatrix} & 1 \begin{bmatrix} 1 & 3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}.$$
Either way, we get
Either way, we get

\[
\begin{array}{cccc}
A_{1,1} & A_{1,2} & A_{2,1} & A_{2,2} \\
A_{2,1} & A_{2,2} & A_{2,1} & A_{2,2} \\
A_{1,1} & A_{1,2} & A_{2,1} & A_{2,2} \\
A_{1,1} & A_{2,1} & A_{2,1} & A_{2,2} \\
\end{array}
\]
References


The End