Solving Toeplitz Systems of Equations and the Importance of Conditioning

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What is a Toeplitz Matrix

- A Toeplitz Matrix or Diagonal Constant Matrix is a \( nxn \) matrix where each of the descending diagonals are constant, where

\[
T_n = \begin{bmatrix}
t_0 & t_{-1} & \cdots & t_{-n+1} \\
t_1 & t_0 & \cdots & t_{-2} \\
\vdots & \ddots & \ddots & \vdots \\
t_{n-1} & t_{n-2} & \cdots & t_0
\end{bmatrix}
\]

- Eigenvectors of Toeplitz matrices are sines and cosines.

- Toeplitz matrices are also related to Fast Fourier Transforms (FFT) and when looking at images and signals processing, Fourier Transforms, Hilbert Spaces, and problems involving trigometric moments.
What is a Toeplitz Matrix

**Definition 1.1**
Let $A$ be an $n \times n$ matrix such that $A$ is persymmetric if it is symmetric about its anti-diagonal.

**Definition 1.2**
Let $A$ be a $n \times n$ matrix such that $A$ is centrosymmetric if it is symmetric about the center.

**Definition 1.3**
Let $A$ be a $n \times n$ matrix. $A$ is bisymmetric if only if $A$ is centrosymmetric and either symmetric or antisymmetric.
What is conditioning? Why does it matter?

The Conditioning Number of a Matrix

\[ \kappa(A) = \| A \| \| A^{-1} \| \geq 1 \]  

- if \( \kappa(A) \) is large than the matrix \( A \) is ill-conditioned
- if \( \kappa(A) \) is small than the matrix \( A \) is well-conditioned
Matrix Norms How do we calculate a Matrix Norm? There are three commonly used norms

1-Norm

Let $A$ be an $m \times n$ matrix. The 1-norm, $\|A\|_1$ is equal to the maximum column sum or for $1 \leq j \leq n$ and $a_j$ is the $j$th column of $A$

$$\|A\|_1 = \max_j \sum_{k=1}^{n} a_{kj}$$ (2)
Matrix Norms

2-Norm

Let $A$ be an $m \times n$ matrix. The 1-norm, $\|A\|_2$ is equal to the largest singular value of $A$

$$\|A\|_2 = \max_i \delta$$ (3)
\[ \|A\|_\infty = \max_i \sum_{k=1}^{m} a_{ik} \] (4)
Outline

1. Toeplitz Matrices
2. Conditioning
3. Matrix Norms
4. Block Gaussian Elimination
5. Large Example
6. Conclusion
Why would we choose block Gaussian elimination compared to other algorithms? What is block Gaussian elimination?
Suppose we have the system \( Tx = b \) where \( T \) is Toeplitz, symmetric and nonsingular. Then partition \( T \)

\[
T \mathbf{x} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
\hat{\mathbf{x}} \\
\check{\mathbf{x}}
\end{bmatrix}
= \begin{bmatrix}
\hat{\mathbf{b}} \\
\check{\mathbf{b}}
\end{bmatrix}
= \mathbf{b}
\]

(5)

where \( \mathbf{x} \) and \( \mathbf{b} \) are \( nx1 \), \( A \) is \( k \times k \), \( B \) is \( k \times (n - k) \), \( C \) is \( (n - k) \times k \), \( D \) is \( (n - k) \times (n - k) \), \( \hat{\mathbf{x}} \) and \( \hat{\mathbf{b}} \) are \( k \times 1 \) and \( \check{\mathbf{x}} \) and \( \check{\mathbf{b}} \) are \( (n - k) \times 1 \).
we then use block Gaussian elimination to break our new partition matrix into an upper and lower triangular matrices

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
CA^{-1} & I
\end{bmatrix}
\begin{bmatrix}
A & B \\
0 & \Delta
\end{bmatrix}
\] (6)

Where \( \Delta = D - CA^{-1}B \), and

\[
\begin{bmatrix}
A & B \\
0 & \Delta
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{x}
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
-CA^{-1} & I
\end{bmatrix}
\begin{bmatrix}
\hat{b} \\
\hat{b}
\end{bmatrix}
= \begin{bmatrix}
\hat{x} \\
\hat{x} - CA^{-1}\hat{b}
\end{bmatrix}
\] (7)
We then solve for $\hat{x}$ and $\check{x}$ by

1. Solving $AX = C$ for $X$, where $X$ is $(n - k) \times k$ matrix
2. Forming $\Delta = D - XB$
3. Forming $\check{c} = \check{b} - X\hat{b}$
4. Solving $\Delta \check{x} = \check{c}$ for $\check{x}$
5. Forming $\hat{c} = \hat{b} - B\check{x}$ and
6. Solving $A\hat{x} = \hat{c}$ for $\hat{x}$.

Though this method is pretty stable there can be problems
The biggest problem with block Gaussian elimination is that even if \( T \) is well-conditioned, \( A \) can be ill-conditioned. There is only one class of matrices that proves that to be true-symmetric, positive-definite matrices, or Hermitian in the complex case.
let us take the 2-norm of both $T$ and $A$

$$\kappa_2(T) = \frac{\sigma_{\text{max}}(T)}{\sigma_{\text{min}}(T)}$$  \hspace{1cm} (8)$$

$$\kappa_2(A) = \frac{\sigma_{\text{max}}(A)}{\sigma_{\text{min}}(A)}$$  \hspace{1cm} (9)$$

where $\sigma_{\text{max}}$ is the largest singular value and $\sigma_{\text{min}}$ is the smallest. Since $T$ and $A$ are symmetric positive definite, $\sigma_{\text{max}}(T) = \lambda_{\text{max}}(T)$, $\sigma_{\text{min}}(T) = \lambda_{\text{min}}(T)$, $\sigma_{\text{max}}(A) = \lambda_{\text{max}}(A)$, $\sigma_{\text{min}}(A) = \lambda_{\text{min}}(A)$, where $\lambda_{\text{max}}$ is the largest eigenvalue and $\lambda_{\text{min}}$ is the smallest.
Cauchy Interlace Theorem

Let $A$ be a symmetric $n \times n$ matrix. Let $B$ an $m \times m$ matrix where $m \leq n$. Let $B$ also be the compression of $A$. If the eigenvalues of $A$ are $\alpha_1 \leq \cdots \leq \alpha_n$, and those of $B$ are $\beta_1 \leq \cdots \leq \beta_j \leq \cdots \leq \beta_m$ then for all $j < m + 1$
From the Cauchy Interlace Theorem we know,

\[ 0 < \lambda_{\text{min}}(T) \leq \lambda_{\text{min}}(A) \leq \lambda_{\text{max}}(A) \leq \lambda_{\text{max}}(T) \]  \hspace{1cm} (10)

Thus,

\[ \kappa_2(A) = \frac{\lambda_{\text{max}}(A)}{\lambda_{\text{min}}(A)} \leq \frac{\lambda_{\text{max}}(T)}{\lambda_{\text{min}}(T)} = \kappa_2(T) \]  \hspace{1cm} (11)

Therefore if \( T \) is well-conditioned then \( A \) is also well-conditioned.
Consider the matrix

\[
T = \begin{bmatrix}
1 & 2 & 0 & -1 & 5 & 8 \\
2 & 1 & 2 & 0 & -1 & 5 \\
0 & 2 & 1 & 2 & 0 & -1 \\
-1 & 0 & 2 & 1 & 2 & 0 \\
5 & -1 & 0 & 2 & 1 & 2 \\
8 & 5 & -1 & 0 & 2 & 1 \\
\end{bmatrix}
\]  

(12)

where \( T \) is symmetric, nonsingular and positive-definite.
Before partitioning the matrix, check the conditioning

\[ \| T \|_1 = 15 \]  \hspace{1cm} (13)

\[ \| T \|_2 \approx 12.822 \]  \hspace{1cm} (14)

\[ \| T \|_{\infty} = 15 \]  \hspace{1cm} (15)

\[ \| T^{-1} \|_1 \approx .284 \]  \hspace{1cm} (16)

\[ \| T^{-1} \|_2 \approx .784 \]  \hspace{1cm} (17)

\[ \| T^{-1} \|_{\infty} \approx .284 \]  \hspace{1cm} (18)
Knowing all three matrix norms, we compute the conditioning numbers

\[ \kappa(T)_1 = \| T \|_1 \| T^{-1} \|_1 = (15)(.284) = 4.26 \] (19)

\[ \kappa(T)_2 = \| T \|_2 \| T^{-1} \|_2 = (12.822)(.784) = 10.05 \] (20)

\[ \kappa(T)_\infty = \| T \|_\infty \| T^{-1} \|_\infty = (15)(.284) = 4.26 \] (21)

Since \( \kappa(T) \) is relatively small then \( T \) is well-conditioned.
Paritition $T$

$$A = \begin{bmatrix}
1 & 2 & 0 \\
2 & 1 & 2 \\
0 & 2 & 1
\end{bmatrix}$$

$$B = \begin{bmatrix}
-1 & 5 & 8 \\
0 & -1 & 5 \\
2 & 0 & -1
\end{bmatrix}$$

$$C = \begin{bmatrix}
-1 & 0 & 2 \\
5 & -1 & 0 \\
8 & 5 & -1
\end{bmatrix}$$

$$D = \begin{bmatrix}
1 & 2 & 0 \\
2 & 1 & 2 \\
0 & 2 & 1
\end{bmatrix}$$
Large Example

\[
\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}
\]

\[
\hat{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}
\]
now we calculate $CA^{-1}$ and $\Delta$

$$CA^{-1} = \begin{bmatrix} \frac{11}{7} & \frac{2}{7} & \frac{10}{7} \\ \frac{13}{7} & \frac{11}{7} & \frac{22}{7} \\ \frac{38}{7} & \frac{9}{7} & \frac{25}{7} \end{bmatrix} \quad (22)$$

$$\Delta = D - CA^{-1}B = \begin{bmatrix} \frac{-24}{7} & \frac{71}{7} & \frac{88}{7} \\ \frac{71}{7} & \frac{-47}{7} & \frac{-167}{7} \\ \frac{88}{7} & \frac{-167}{7} & \frac{-367}{7} \end{bmatrix} \quad (23)$$
Now we solve for $x$

$$\tilde{c} = \tilde{b} - CA^{-1}\hat{b} = \begin{bmatrix} \frac{19}{7} \\ \frac{67}{7} \\ \frac{65}{7} \end{bmatrix} \quad (24)$$

$$\tilde{x} = \Delta^{-1}\tilde{c} = \begin{bmatrix} -\frac{9418}{7807} \\ -\frac{21}{7807} \\ -\frac{866}{7807} \end{bmatrix} \approx \begin{bmatrix} -1.2063532727 \\ -0.00268989368515 \\ -0.110926091969 \end{bmatrix} \quad (25)$$
Since we have $\tilde{x}$, we can finally solve for $\hat{x}$

$$\hat{x} = A^{-1}(\hat{b} - B\tilde{x}) = \begin{bmatrix} -\frac{22}{7807} \\ \frac{2722}{7807} \\ \frac{7807}{4719} \\ \frac{7807}{7807} \end{bmatrix} \approx \begin{bmatrix} -0.00281798386064 \\ 0.348661457666 \\ 0.604457538107 \end{bmatrix}$$

(26)
Large Example

\[ x = \begin{bmatrix} \hat{x} \\ \check{x} \end{bmatrix} = \begin{bmatrix} -0.00281798386064 \\ 0.348661457666 \\ 0.604457538107 \\ 1.2063532727 \\ -0.00268989368515 \\ -0.110926091969 \end{bmatrix} \] (27)
Final Thoughts

- Block Gaussian Elimination uses $O(n^2)$ flops while preserving Toeplitz structure.
- The block matrix $A$ must be proven to be well-conditioned or else it can ruin your solution(s).

