Tournament Matrices

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1 Introduction

A round-robin tournament consists of n teams (or players) that each play against every other team once. By using a 0 to represent a loss and a 1 to represent a win, we can use a matrix to record the outcomes of some round-robin tournament. This matrix is called a *tournament matrix*. If we label the teams recorded in a tournament matrix A of size n as $\{x_1, x_2, ..., x_n\}$, then we can say that team x_i beats team x_j if there is a 1 in the entry $[A]_{ij}$. This means that there is a corresponding 0 in the entry $[A]_{ji}$ because team x_j lost to team x_i. Therefore,

$$[A]_{ij} + [A]_{ji} = 1 \text{ for } 1 \leq i < j \leq n.$$

Also, teams cannot play themselves, so

$$[A]_{ii} = 0 \text{ for } 1 \leq i \leq n.$$

These two properties of tournament matrices can be conjoined into the equivalent statement

$$A + A^T = J_n - I_n$$

where J_n is the $n \times n$ matrix of all 1's and I_n is the Identity Matrix. Variations of this equality will be used to describe nonbinary types of tournament matrices in the subsequent section.

Here is an example of a tournament matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

A tournament matrix is the adjacency matrix of a *tournament*. Denoted as $T(A)$, a tournament for the tournament matrix A is a digraph obtained from the complete graph K_n by assigning a direction to each edge. The vertices of a tournament are the teams playing in the round-robin tournament and the orientation to the edges is dependent upon the winner between two teams. If team x_i beats team x_j, then there will be a directed edge pointing from x_i to x_j, or similarly a 1 in the (i, j) entry of the corresponding tournament matrix.

If the sole purpose of a tournament matrix is simply to represent the outcomes of a round-robin tournament, it does not seem all that useful as outcomes can be represented in a variety of ways, such as a graph like stated above. However, a tournament matrix has properties that can give rankings to the teams playing in the corresponding round-robin tournament. These rankings can then be used to determine which team was the strongest in the round-robin tournament and can furthermore be used to make an educated assumption on who may win in a different tournament. Before getting into rankings, though, some more information on tournament matrices is needed.

2 Landau's Theorem and Nonbinary Types

To show that a tournament matrix exists, we have Landau's Theorem. But first, some definitions.

Given A, let $R = (r_1, r_2, ..., r_n)$ be the *row sum vector* of matrix A, or, in other words, the vector containing the sums of each row of matrix A in subsequent order. R can also be written as $R = R(A) = A1_n$, where 1_n is the size n column vector of all 1's. Let $S = (s_1, s_2, ..., s_n)$ be the column sum vector. For tournament matrices, there is no loss of generality if R is assumed to be nondecreasing and, as a result, S is assumed to be nonincreasing. The sum of all elements in a generalized tournament matrix of size n is

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} s_i = \sum_{i=1}^{n}\sum_{j=1}^{n} [A]_{ij} = \binom{n}{2}$$

The row sum r_i of a tournament matrix A is the number of wins that team x_i obtained in the round-robin tournament. For this reason, r_i is also referred to as the *score* of team x_i and the row sum vector R is called the *score vector*. The class of all tournament matrices with row sum vector R is denoted by $\mathcal{T}(R)$.

Landau characterized the score vectors of tournaments to prove existence of tournament matrices in $\mathcal{T}(R)$ in the following theorem:

Theorem 2.1 *Let n be a positive integer and let $R = (r_1, r_2, ..., r_n)$ be a nondecreasing, nonnegative integral vector. Then $\mathcal{T}(R)$ is nonempty if and only if*

$$\sum_{i=1}^{k} r_i \geq \binom{k}{2} \text{ for } 1 \leq k \leq n,$$

with equality when $k = n$.

There are multiple nonbinary types of tournament matrices. This paper will only discuss a few. A *generalized tournament matrix* is a tournament matrix where values of the entries are between 0 and 1 inclusive. All properties of a tournament matrix stated above apply to a generalized tournament matrix, including Landau's Theorem. In a generalized tournament matrix A, $[A]_{ij}$ can be interpreted as the probability that team x_i will beat team x_j; the row sum vector then gives the expected number of wins of each team. To determine this probability, we will use the ranking system discussed in the final sections. Finally, the class of all generalized tournament matrices with row sum vector R is denoted by $\mathcal{T}^g(R)$.

The rank of a generalized tournament matrix is given in the following theorem.

Theorem 2.2 *Let A be a generalized tournament matrix of size n. Then the rank of A is at least $n - 1$.*

Proof By definition of generalized tournament matrices, $A + A^T = J_n - I_n$. Append 1_n, the size n column vector of all 1's, to A and call that matrix B. Suppose that for some real column vector x of size n, $x^T B = 0$. Then $x^T 1_n = 0$ and $x^T A = 0$, and hence $x^T J_n = 0$ and $A^T x = 0$, respectively. Consider,

$$x^T J_n x - x^t x = x^T (J_n - I_n)x = x^T (A + A^T)x = (x^T A)x + x^T (A^t x) = 0.$$

Since $x^T J_n x = (x^T J_n)x = 0$, $-x^T x$ must be equal to 0. And consequently, x is the zero vector. $x^T B = B^t x = 0$, so the nullity of B^T is 0, which means that the rank of B^T is n. The rank of a matrix is equal to the rank of the matrix transposed, so the rank of B is also n. Therefore, we conclude that the rank of A is at least $n-1$.

We can consider a round-robin tournament where each team plays p games against every other team. The representative matrix of this kind of tournament is called a *p-tournament matrix*. A p-tournament matrix A of size n satisfies the equality

$$[A]_{ij} + [A]_{ji} = p \text{ for } (1 \le i < j \le n),$$

or equivalently,

$$A + A^T = p(J_n - I_n).$$

Therefore, the sum of all elements of a p-tournament matrix is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} [A]_{ij} = \sum_{1 \le i < j \le n} [A]_{ij} + [A]_{ji} = \sum_{1 \le i < j \le n} p = p\binom{n}{2}$$

So element $[A]_{ij}$ of a p-tournament matrix represents the number of games in which team x_i defeats team x_j. The class of all p-tournament matrices with row sum vector R is denoted as $\mathcal{T}(R; p)$.

Landau's Theorem can easily be manipulated to account for the existence of p-tournament matrices:

Theorem 2.3 *Let p and n be positive integers. Let $R = (r_1, r_2, ..., r_n)$ be a nondecreasing, nonnegative integral vector. Then $\mathcal{T}(R; p)$ is nonempty if and only if*

$$\sum_{i=1}^{k} r_i \le p\binom{k}{2} \text{ for } 1 \le k \le n,$$

with equality when $k = n$

We can go even more general and define a *P-tournament matrix*, where P is a nonnegative integral upper-triangular matrix with 0's on the main diagonal and $[P]_{ij}$ is the number of games to be played between teams x_i and x_j. Therefore, a P-tournament matrix satisfies the equalities

$$[A]_{ij} + [A]_{ji} = [P]_{ij} \text{ for } 1 \le i < j \le n$$

and

$$A + A^T = P + P^T.$$

The sum of all entries of a P-tournament matrix is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} [A]_{ij} = \sum_{1 \le i < j \le n} [A]_{ij} + [A]_{ji} = \sum_{1 \le i < j \le n} [P]_{ij}$$

and the class of all P-tournament matrices with row sum vector R is denoted by $\mathcal{T}(R;P)$.

Once again, Landau's Theorem can be manipulated to support the existence of P-tournament matrices:

Theorem 2.4 *Let n be a positive integer and let P be a nonnegative integral upper-triangular matrix of size n with 0's on the main diagonal. Let $R = (r_1, r_2, ..., r_n)$ be a nonnegative integral vector. Then $\mathcal{T}(R;P)$ is nonempty if and only if*

$$\sum_{i \in K} r_i \geq \sum_{i \in K} \sum_{j \in K} [P]_{ij} \text{ for } K \subset \{1, 2, ..., n\},$$

with equality when $K = \{1, 2, ..., n\}$.

3 Regular and Near-Regular Tournament Matrices

Regular tournament matrices and near-regular tournament matrices are tournament matrices that are dependent upon the score vector. A tournament matrix A of size n is a *regular tournament matrix* if n is odd and every entry of its score vector is $(n-1)/2$. For example,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is a regular tournament matrix. A tournament matrix B of size n is *near-regular tournament matrix* if n is even and half of the elements of the score vector are $n/2$ and the other half are $(n-2)/2$. Here is an example of a near-regular tournament matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Both regular tournament matrices and near-regular tournament matrices are nonsingular and irreducible. A square matrix A is *irreducible* if there exists no permutation matrix P such that PAP^T is a block upper-triangular matrix with two or more square diagonal blocks. Both also have their own unique properties. Regular tournament matrices are normal, unitarily diagonalizable, and have a *spectral radius*, the largest sign-independent eigenvalue denoted as ρ, of $(n-1)/2$ that satisfy $A1_n = \rho 1_n$. Also, given a score vector R of size n where n is odd, the tournament matrices with the largest spectral radius in $\mathcal{T}(R)$ are the regular tournament matrices.

Near-regular tournament matrices' properties are not nearly as nice as regular tournament matrices. For example, there is no easy way to determine the spectral radius of near-regular tournament matrix; all that is known is that the spectral radius of a near-regular tournament matrix exceeds $(n-2)/2$. Also, it is not known when near-regular tournament matrices are diagonalizable, though it is known that they are never unitarily diagonalizable.

Despite their unsatisfying properties, there is a nice method to construct near-regular tournament matrices using tournament matrices.

Theorem 3.1 *Let A be any $n \times n$ tournament matrix. Then,*

$$M_A = \begin{bmatrix} A & A^T \\ A^T + I_n & A \end{bmatrix}$$

is a $2n \times 2n$ near-regular tournament matrix.

Proof Since $A + A^T = J_n - I_n$, the first n rows of M_A have row sum $n-1$ and the last n rows of M_A have row sum n. So the score vector of M_A is

$$M_A 1_{2n} = \begin{bmatrix} (n-1)1_n \\ n1_n \end{bmatrix}.$$

Therefore, by definition, M_A is a near-regular tournament matrix.

A more specific near-tournament matrix with this form is one where A is the $n \times n$ strictly lower triangular tournament matrix. This near-tournament matrix is called the Brualdi-Li Tournament Matrix.

4 Brualdi-Li Tournament Matrices

A more formal definition of the Brualdi-Li Matrix is as follows: the *Brualdi-Li* matrix is a near-regular tournament matrix of size $2m$ defined as

$$\mathcal{B}_{2n} = \begin{bmatrix} L_n & L_n^T \\ I_n + L_n^T & L_n \end{bmatrix}$$

where L_m is the strictly lower $m \times m$ tournament matrix. Here is an example of a Brualdi-Li Matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The Brualdi-Li Tournament Matrix $\mathcal{B}_{2m}$ has the largest spectral radius among all tournament matrices of even order. If a tournament matrix A shares the largest spectral radius with $\mathcal{B}_{2m}$, then A is permutationally similar to the $\mathcal{B}_{2m}$, which means $PAP^T = \mathcal{B}_{2m}$ where P is a permutation matrix.

Theorem 4.1 *Let $\mathcal{B}_{2m}$ be the $2m \times 2m$ Brualdi-Li matrix. Then $\rho(\mathcal{B}_{2m}) \geq \rho(T)$ for every $2m \times 2m$ tournament T; in the case of equality, T is permutationally similar to $\mathcal{B}_{2m}$.*

Regular tournaments have much nicer properties compared to near-regular tournaments, such as being unitarily diagonalizable, having 1_n as an eigenvector, and having a common spectral radius among regular tournaments of the same size. With the Brualdi-Li matrix, however, near-regular tournaments are given redemption with nice properties such as the theorem stated above. This theorem also tells us that among all $2m \times 2m$ tournament matrices (for m sufficiently large), the tournament matrices with maximum spectral radius are near-regular tournament matrices. On top of that, these matrices are permutationally similar to $\mathcal{B}_{2m}$. Brualdi-Li matrices are also diagonalizable, though not unitarily.

5 Upsets

Recall that a tournament matrix A with score vector $R = (r_1, r_2, ..., r_n)$ in $\mathcal{T}(R)$ is a binary matrix that is the adjacency matrix of a digraph called a tournament, $T(A)$, with a vertex set $\{x_1, x_2, ..., x_n\}$. A can be used to represent the outcomes of a round-robin tournament amongst teams represented by the vertices. A victory by team x_i against team x_j is portrayed by a 1 in entry (i, j) of A or equivalently an edge from x_i to x_j in $T(A)$. Since R is a nondecreasing integer vector, $r_1 \leq r_2 \leq ... \leq r_n$, which means that if $i < j$, then the record of team x_i is no better than that of x_j. In other words, x_i is *inferior* to x_j and x_j is *superior* to x_i. If $r_i = r_j$, then superiority is determined by the order the teams are listed (the latter being the superior). An *upset* occurs whenever a superior team is defeated by an inferior team; therefore, an upset is symbolized by a 1 above the main diagonal of A. The total number of upsets is denoted as

$$v(A) = \sum_{i<j} [A]_{ij}.$$

$v(A) = 0$ if and only if A has 0's on and above its main diagonal.

Theorem 5.1 *Let A be a matrix in $\mathcal{T}(R)$ with the minimal number of upsets. Then*

$$[A]_{21} = [A]_{32} = ... = [A]_{n,n-1} = 1 \text{ and } [A]_{12} = [A]_{23} = ... = [A]_{n-1,n} = 0.$$

So in other words, a matrix in $\mathcal{T}(R)$ that has the minimal number of upsets has 1's on the sub diagonal and 0's on the super diagonal.

The following theorem describes the actual number of minimal upsets of the matrices stated in the previous theorem:

Theorem 5.2 *Let R be a nondecreasing integral vector satsifying Landau's theorem. Then the matrices in $\mathcal{T}(R)$ with the minimal number of upsets has*

$$\tilde{v}(R) = \sum_{i=1}^{n} |r_i(i-1)|$$

upsets.

6 Ranking

We are almost ready to begin ranking teams of a tournament matrix. To compare the teams in a tournament matrix, it makes sense to look at the *score* of each team, or the total amount of wins each has, and simply rank the teams by those numbers with 1st rank awarded to the team with the most wins. However, there is a better way to rank teams. But first, a couple theorems.

Theorem 6.1 *Let n be a positive integer and let $R = (r_1, r_2, ...r_n)$ be a nondecreasing, nonnegative integral vector. The following are equivalent.*

1. *There exists an irreducible matrix in $\mathcal{T}(R)$.*
2. *$\mathcal{T}(R)$ is nonempty and every matrix in $\mathcal{T}(R)$ is irreducible.*
3. *$\sum_{i=1}^{k} r_i \geq \binom{k}{2}$ for $1 \leq k \leq n$, with equality if and only if $k = n$.*

So if there is one irreducible matrix in $\mathcal{T}(R)$ that satisfies Landau's Theorem, then every matrix in $\mathcal{T}(R)$ is irreducible. Therefore, we focus our attention on irreducible tournament matrices. The following theorem is called the Perron-Frobenius Theorem.

Theorem 6.2 *Let M be a nonnegative, irreducible matrix. Then the spectral radius of M, $\rho(M)$, is a unique, positive eigenvalue for M, and there is an entrywise positive eigenvector v. Such a vector v is called the Perron vector for ρ.*

Let us say we have a tournament matrix A. Now suppose we measure the strength of team x_i by computing the sum of the scores of the team that x_i beats: $\sum s_j$, where s_j is the score vector of the team defeated by x_i. After all, a strong team should be able to beat teams that have also beaten a lot of other teams. Since x_i beats x_j when $[A]_{ij} = 1$, the measure of strength is

$$\sum_{j=1}^{n} [A]_{ij} s_j = \sum_{j=1}^{n} ([A]_{ij} \sum_{k=1}^{n} [A]_{jk}] = \sum_{k=1}^{n} \sum_{j=1}^{n} [A]_{ij} [A]_{jk}.$$

This is the sum of all entries in the i^{th} row of A^2. In other words, $A^2 1_n$ is the vector whose i^{th} entry is the sum of the scores of all teams defeated by x_i. We can continue this process by taking greater and greater powers of A up to some arbitrary postive integer k. For example, the i^{th} entry of $A^3 1_n$ is the sum of the sum of the scores of teams defeated by team x_i. As k increases, $A^k 1_n$ becomes more sensitive to the relationships among the teams and therefore provides a better sense of rank to each team.

We will now include the the Perron-Frobenius theorem. Let A be an irreducible tournament matrix. For $1 \leq k \leq \infty$, let $l_k = ||A^k 1_n||$. Then the sequence of nonnegative vectors $l_1^{-1} A 1_n, l_2^{-1} A^2 1 - n, l_3^{-1} A^3 1_n, ..., l_k^{-1} A^k 1_n, ...$ converges to the Perron vector v for A, or in other words, $v = \lim_{k \to \infty} l_k^{-1} A^k 1_n$. This is so because this process is basically the method of Power Iteration, which is the method of finding an eigenvector for the spectral radius. And as given in the theorem, the Perron vector v is the eigenvector for the spectral radius of an irreducible matrix. Thus, the relative sizes of the entries in the Perron vector v for

A provides a much better way to rank the teams in a tournament. This approach is called the *Kendall-Wei Ranking*. Interestingly enough, $s_i > s_j$ does not necessarily mean $v_i > v_j$, though for Brualdi-Li Matrices it does, as will be demonstrated in a bit.

Another method of ranking comes from Ramanujacharyula, who suggests a strength to weakness ratio to rank the teams in a round-robin tournament. Suppose that A is an irreducible tournament matrix. The strength part of the ratio comes from the Perron vector v of A, specifically the right Perron vector which is the vector that satisfies the equality $Av = \rho(A)v$ and the one we just showed how to calculate above. The weakness then comes from the left Perron vector w of A which satisfies the equality $w^T A = \rho(A)w^T$. To calculate the left Perron vector, you just use the power method again: the left Perron vector of an irreducible tournament matrix A is $w = \lim_{k\to\infty} b_k^{-1} 1_n^T A^k$ where $b_k = ||1_n^T A^k||$. So by using these ratios, we determine that team x_i is stronger than team x_j if $v_i/w_i > v_j/w_j$.

Brualdi-Li Tournament Matrices have nice properties when it comes to rankings. In relation to the Kendall-Wei ranking method, the team x_{2n} in the round-robin tournament represented by $\mathcal{B}_{2n}$, where $n \geq 2$, ranks the highest, followed in decreasing rank by teams $2n-1, 2n-2, ...n+1, 1, 2, ..., n$. This is shown in the following theorem:

Theorem 6.3 *Let $v \in \mathcal{R}^n$ and $w \in \mathcal{R}^n$ for $n \geq 2$ so that $x = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathcal{R}^{2n}$ is the Perron vector of $\mathcal{B}_{2n}$. Then*

$$v_n < v_{n-1} < ... < v_1 < w_1 < w_2 < ... < w_n$$

In addition, among all tournaments with an even number of teams, the tournament corresponding to the Brualdi-Li matrix has the most well-matched teams in that there is a minimal variation in their Kendall-Wei ranks.

In relation to the Ramanujacharyula ranking scheme, the teams of the round-robin tournament represented by the Brualdi-Li Matrix $\mathcal{B}_{2n}$ are related by the following theorem.

Theorem 6.4 *Let v be the right Perron vector of $\mathcal{B}_{2n}$ and let w be the left Perron vector of $\mathcal{B}_{2n}$. Then, we have the following interlacing relationships:*

$$\frac{v_n}{w_n} < \frac{v_1}{w_1} < \frac{v_{n-1}}{w_{n-1}} < \frac{v_2}{w_2} < \frac{v_{n-2}}{w_{n-2}} < ... < \frac{v_{n/2}}{w_{n/2}} < 1,$$
$$1 < \frac{v_{2n-n/2+1}}{w_{2n-n/2+1}} < ... < \frac{v_{n+3}}{w_{n+3}} < \frac{v_{2n-1}}{w_{2n-1}} < \frac{v_{2n}}{w_{2n}} < \frac{v_{n+1}}{w_{n+1}}$$

where $n/2$ is rounded up if n is odd.

Notice that the Kendall-Wei ranking scheme and the Ramanujacharyula ranking scheme both agree with ranking via row sum vectors for the Brualdi-Li Matrix and in fact for near-regular tournament matrices in general. The teams in $\mathcal{B}_{2n}$ whose row sums are $n-2$ all have lower rankings than the teams whose row sums are n.

Can we apply ranking to nonbinary tournament matrices? The answer is yes. For p-tournament matrices, the same process of determining the Perron vector can be used. This is allowed because the entries of the Perron vector are relative to eachother so as long as each team plays the same amount of games, the power method used to determine the Perron vector will apply. However, this means there is a problem with P-tournament matrices since each team does not necessarily play the same amount of games. It is not to say you cannot rank a P-tournament matrix; it will simply require more precautions, that we will not get into, so that there is no bias towards teams that have played more games.

Recall that generalized tournament matrices represent the probabilities that a team will defeat another team. These probabilities can be determined via the Perron vector. Let v be the Perron vector of a tournament matrix A. Then the probability that team x_i defeats team x_j is

$$\pi_{ij} = \frac{v_i}{v_i + v_j}.$$

A generalized tournament matrix G would then be defined as

$$[G]_{ij} = \pi_{ij}$$

So the probabilities are determined after the outcomes have already happened. This means that generalized tournament matrices are only useful when you already know the rankings. Since rankings of tournament matrices are relative to the teams in the given tournament, these rankings cannot be used to compare teams between two different tournaments. Therefore, generalized tournament matrices can only be used in a situation where the same teams in a tournament play in another tournament. If you divide a p-tournament into p single tournaments, you can determine the probabilities of the current tournament being played by using the rankings of the previous tournaments.

7 One Big Example

We conclude with one big example that demonstrates the main concepts in this paper. Consider the Brualdi-Li Tournament Matrix $\mathcal{B}_{12}$:

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}$$

The right Perron vector is

$$v = \lim_{k\to\infty} \frac{\mathcal{B}_{12}^k 1_{12}}{||\mathcal{B}_{12}^k 1_{12}||} =$$
$$\begin{bmatrix} .282 & .279 & .275 & .269 & .261 & .250 & .296 & .298 & .302 & .307 & .313 & .323 \end{bmatrix}$$

and the left Perron vector is

$$w = \lim_{k\to\infty} \frac{1_{12}^T \mathcal{B}_{12}^k}{||1_{12}^T \mathcal{B}_{12}^k||} =$$
$$\begin{bmatrix} .323 & .313 & .307 & .302 & .298 & .296 & .250 & .261 & .269 & .275 & .279 & .282 \end{bmatrix}$$

with decimals rounded to three significant figures. The ratios given by Theorem 6.4 are

$$\frac{v_6}{w_6} = .845 < \frac{v_1}{w_1} = .873 < \frac{v_5}{w_5} = .876 < \frac{v_2}{w_2} = .890 < \frac{v_4}{w_4} = .891 < \frac{v_3}{w_3} = .896 < 1$$
$$1 < \frac{v_{10}}{w_{10}} = 1.116 < \frac{v_9}{w_9} = 1.122 < \frac{v_{11}}{w_{11}} = 1.123 < \frac{v_8}{w_8} = 1.142 < \frac{v_{12}}{w_{12}} = 1.145 < \frac{v_7}{w_7} = 1.184$$

According to the Kendall-Wei ranking, the teams $1, 2, ..., 12$ rank as follows:

$$12, 11, 10, 9, 8, 7, 1, 2, 3, 4, 5, 6$$

whereas the Ramanucharyula scheme ranks the teams as

$$7, 12, 8, 11, 9, 10, 3, 4, 2, 5, 1, 6.$$

So which one is better? Well it depends on what you are looking for in these ranks. The Kendall-Wei ranking method focuses entirely on the strength of a team whereas the Ramanucharyula ranking scheme includes weaknesses as well. Notice, however, that the top 6 teams in both lists are the bottom 6 rows of $\mathcal{B}_{12}$ which have row sums 6 and the bottom 6 teams are the top 6 rows of $\mathcal{B}_{12}$ which have row sums 5.

If these same teams were to play again, we can now calculate the probabilties that one team will defeat another and store them in a generalized tournament matrix. Recall that the probability team x_i will defeat team x_j is $\pi_{ij} = \dfrac{v_i}{v_i + v_j}$. Here are the results:

$$\begin{bmatrix}
0 & .503 & .506 & .512 & .519 & .530 & .488 & .486 & .483 & .479 & .474 & .466 \\
.497 & 0 & .504 & .509 & .517 & .527 & .485 & .484 & .480 & .476 & .471 & .463 \\
.494 & .496 & 0 & .506 & .513 & .524 & .482 & .480 & .477 & .473 & .468 & .460 \\
.488 & .491 & .494 & 0 & .507 & .518 & .476 & .474 & .471 & .467 & .462 & .454 \\
.481 & .483 & .487 & .493 & 0 & .511 & .469 & .467 & .464 & .460 & .455 & .447 \\
.470 & .473 & .476 & .482 & .489 & 0 & .458 & .456 & .453 & .449 & .444 & .436 \\
.512 & .515 & .518 & .524 & .531 & .542 & 0 & .498 & .495 & .491 & .486 & .478 \\
.514 & .516 & .520 & .526 & .533 & .544 & .502 & 0 & .497 & .493 & .488 & .480 \\
.517 & .520 & .523 & .529 & .536 & .547 & .505 & .503 & 0 & .496 & .491 & .483 \\
.521 & .524 & .527 & .533 & .540 & .551 & .509 & .507 & .504 & 0 & .495 & .487 \\
.526 & .529 & .532 & .538 & .545 & .556 & .514 & .512 & .509 & .505 & 0 & .492 \\
.534 & .537 & .540 & .546 & .553 & .564 & .522 & .520 & .517 & .513 & .508 & 0
\end{bmatrix}$$

Notice how there is only a small range of numbers in this matrix. That is because of all tournaments with an even number of teams, the tournament corresponding to the Brualdi-Li matrix has the most well-matched teams in the sense of minimizing the variation in the entries of its Perron vector. So in other words, this is as close as it gets in a 12 team round-robin tournament.

8 Conclusion

The Kendall-Wei ranking scheme and the Ramanucharyula ranking scheme are not the only ranking schemes out there. For example, the idea of minimizing the amount of upsets in a tournament can be used to determine ranking. Though the Kendall-Wei and Ramanucharyula ranking schemes are the most common ranking schemes, a weakness that lies in these two schemes is that the strength of schedule is quite important. If a strong team plays mostly weak teams, then the strong team cannot earn a high ranking. Similarly, a team that loses most of its games but does reasonably well against strong teams can still earn a high ranking. A solution to this problem goes beyond the scope of linear algebra and into nonlinear areas of mathematics. But regardless, the Kendall-Wei and Ramanucharyula ranking schemes are reasonable estimates for rank depending on what you consider important in ranking a team or player. So when the next superbowl comes around and you are betting on a team to win, use the Kendall-Wei and Ramanucharyula ranking schemes to give yourself the advantage.

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