Tournament Matrices

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Tournaments

- Digraph that represents the outcome of a round-robin tournament
Graph Theory

Tournaments

- Digraph that represents the outcome of a round-robin tournament
- Vertices are teams
- Edges denote the victor between two teams
- $K_n$ with direction
Graph Theory

Example
Tournament Matrices

▸ Adjacency matrix of a tournament
Tournament Matrices

- Adjacency matrix of a tournament
- 1’s represent wins, 0’s represent losses
Properties

Let $A$ be a tournament matrix of size $n \times n$

- $[A]_{ii} = 0$ for $1 \leq i \leq n$
- $[A]_{ij} + [A]_{ji} = 1$ for $1 \leq i < j \leq n$
- $A + A^T = J_n - I_n$
- $\sum_{i=1}^{n} \sum_{j=1}^{n} [A]_{ij} = \binom{n}{2}$
Matrix Form

Row and Column Sums

- Row sum vector \( R = (r_1, r_2, \ldots, r_n) \) where \( r_i = \sum_{j=1}^{n} [A]_{ij} \)
  - \( r_i \) represents the number of wins team \( i \) has
  - Also known as the score vector
  - \( R = R(A) = Aj_n \)
Row and Column Sums

- **Row sum vector** \( R = (r_1, r_2, \ldots, r_n) \) where \( r_i = \sum_{j=1}^{n} [A]_{ij} \)
  - \( r_i \) represents the number of wins team \( i \) has
  - Also known as the score vector
  - \( R = R(A) = A j_n \)

- **Column sum vector** \( S = (s_1, s_2, \ldots, s_n) \) where \( s_i = \sum_{i=1}^{n} [A]_{ij} \)
  - \( s_i \) represents the number of losses team \( i \) has
## Example

\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Introduction

**Types**

- Perron-Frobenius

**Ranking**

**Big Example**

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**Generalized Tournament Matrices**

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**Generalized Tournament Matrices**

- Tournament matrix where the values of the entries are 0 and 1 inclusive.
- Entries are the probabilities that one team will defeat another.
Generalized Tournament Matrices

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- Entries are the probabilities that one team will defeat another
Regular Tournament Matrices

A tournament matrix $A$ of size $n$ with score vector $R$ is a regular tournament matrix if

- $n$ is odd
- Every entry of $R$ is $(n - 1)/2$

Example:

$$\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}$$
Properties

- Nonsingular
- Irreducible ($PAP^T \neq$ block upper-triangular matrix)
Regular Tournament Matrices

Properties

- Nonsingular
- Irreducible ($PAP^T \neq$ block upper-triangular matrix)
- Normal ($AA^* = A^*A$)
- Unitarily Diagonalizable ($UAU^* = \text{diagonal matrix}$)
Regular Tournament Matrices

Properties

- Nonsingular
- Irreducible ($PAP^T \neq \text{block upper-triangular matrix}$)
- Normal ($AA^* = A^*A$)
- Unitarily Diagonalizable ($UAU^* = \text{diagonal matrix}$)
- Spectral radius $\rho = \rho(A) = (n - 1)/2$
- $A_{jn} = \rho j_n$
- Tournament matrices of size $n$ where $n$ is odd with the largest spectral radius are regular
Near-Regular Tournament Matrices

A tournament matrix $A$ of size $n$ with score vector $R$ is a near-regular tournament matrix if

- $n$ is even
- Half the entries of $R$ are $(n - 2)/2$ and the other half are $n/2$

Example:

$$
\begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
\end{bmatrix}
$$
Near-Regular Tournament Matrices

Construction

Theorem

Let $A$ be any $n \times n$ tournament matrix. Then, $M = [AA^T + I_n]$ is a $2n \times 2n$ near-regular tournament matrix.
Near-Regular Tournament Matrices

Construction

**Theorem**

Let $A$ be any $n \times n$ tournament matrix. Then,

$$M_A = \begin{bmatrix} A & A^T \\ A^T + I_n & A \end{bmatrix}$$

is a $2n \times 2n$ near-regular tournament matrix.
Proof.
Since \( A + A^T = J_n - I_n \), the first \( n \) rows of \( M_A \) have row sum \( n - 1 \) and the last \( n \) rows of \( M_A \) have row sum \( n \). So the score vector of \( M_A \) is

\[
M_A J_{2n} = \begin{bmatrix} (n - 1) j_n \\ n j_n \end{bmatrix}.
\]

Therefore, by definition, \( M_A \) is a near-regular tournament matrix.
Near-Regular Tournament Matrices

**Brualdi-Li Matrix**

Near-regular tournament matrix of size $2m$ defined as

$$B_{2m} = \begin{bmatrix} L_m & L_m^T \\ L_m^T + I_m & L_m \end{bmatrix}$$
Near-regular tournament matrix of size $2m$ defined as

$$B_{2m} = \begin{bmatrix} L_m & L_m^T \\ L_m^T + I_m & L_m \end{bmatrix}$$

Properties:

- $\rho(B_{2m}) \geq \rho(A)$ for every $2m \times 2m$ tournament matrix $A$
- If $\rho(B_{2m}) = \rho(A)$, $PAP^T = B_{2m}$ where $P$ is some permutation matrix
- Diagonalizable, though not unitarily
- First $m$ entries of score vector are $n - 1$. Last $m$ entries are $m$. 

Brualdi-Li Matrix
Perron-Frobenius Theorem

*Theorem*

Let $M$ be a nonnegative, irreducible matrix. Then the spectral radius of $M$, $\rho(M)$, is a unique, positive eigenvalue for $M$, and there is an entrywise positive eigenvector $v$. Such a vector $v$ is called the Perron vector for $\rho$. 

Tournament Matrices
Kendall-Wei Ranking

Let $A$ be a tournament matrix of size $n$. 

Suppose strength of team $i$ is the sum of the scores that team $i$ beats: 

$$n \sum_{j=1}^{n} [A]_{ij} s_j$$

where $s_j$ is the score of team $j$ defeated by team $i$.

This is the sum of all entries in the $i$th row of $A^2$.

$n$ is the vector whose $i$th entry is the sum of the scores of all teams defeated by team $i$.

Continue process up to $A^k j n$ where $k$ is an arbitrary positive integer.

$$\lim_{k \to \infty} A^k j n \quad ||A^k j n|| = \text{Perron vector (Power Method)}$$
Kendall-Wei Ranking

Let $A$ be a tournament matrix of size $n$

- Suppose strength of team $i$ is the sum of the scores that team $i$ beats: $\sum_{j=1}^{n} [A]_{ij} s_j$ where $s_j$ is the score of team $j$ defeated by team $i$. 

Perron-Frobenius Ranking

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Perron vector $v$ (Power Method)
Kendall-Wei Ranking

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- $\sum_{j=1}^{n} [A]_{ij} s_j = \sum_{j=1}^{n} (\sum_{k=1}^{n} [A]_{jk}) = \sum_{k=1}^{n} \sum_{j=1}^{n} [A]_{ij} [A]_{jk}$
Kendall-Wei Ranking

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- $\sum_{j=1}^{n} [A]_{ij} s_j = \sum_{j=1}^{n} ([A]_{ij} \sum_{k=1}^{n} [A]_{jk}) = \sum_{k=1}^{n} \sum_{j=1}^{n} [A]_{ij} [A]_{jk}$

- This is the sum of all entries in the $i^{th}$ row of $A^2 \rightarrow A^2 j_n$ is the vector whose $i^{th}$ entry is the sum of the scores of all teams defeated by team $i$
Kendall-Wei Ranking

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- This is the sum of all entries in the $i^{th}$ row of $A^2 \rightarrow A^2 j_n$ is the vector whose $i^{th}$ entry is the sum of the scores of all teams defeated by team $i$.

- Continue process up to $A^k j_n$ where $k$ is an arbitrary positive integer pause

- $\lim_{k \to \infty} \frac{A^k j_n}{\|A^k j_n\|} = \text{Perron vector } \nu$ (Power Method)
Ramanujacharyula Ranking
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Let $A$ be a tournament matrix of size $n$

- Strength to weakness ratio
Ramanujacharyula Ranking

Let $A$ be a tournament matrix of size $n$

- Strength to weakness ratio
- Strength determined by right Perron vector $v$ ($Av = \rho v$)
- Weakness determined by left Perron vector $w$ ($w^T A = \rho w^T$)
- $w = \lim_{k \to \infty} \frac{j_n^T A^k}{\|j_n^T A^k\|}$
Ramanujacharyula Ranking

Let $A$ be a tournament matrix of size $n$

- Strength to weakness ratio
- Strength determined by right Perron vector $v$ ($Av = \rho v$)
- Weakness determined by left Perron vector $w$ ($w^TA = \rho w^T$)

\[
\begin{align*}
&\lim_{k \to \infty} \frac{j_n^TA^k}{\|j_n^TA^k\|} \\
&\text{Team } i \text{ is stronger than team } j \text{ if } v_i/w_i > v_j/w_j.
\end{align*}
\]
Let $B_{2m}$ be the Brualdi-Li matrix of size $2m$ with right Perron vector $v$ and left Perron vector $w$

- **Kendall-Wei Ranking:**
  \[
  v_{2m} < v_{2m-1} < v_{2m-2} < \ldots < v_{m+1} < v_1 < v_2 < \ldots < v_m
  \]

- **Ramanujacharyula Ranking:**
  \[
  \frac{v_m}{w_m} < \frac{v_1}{w_1} < \frac{v_{m-1}}{w_{m-1}} < \frac{v_2}{w_2} < \frac{v_{m-2}}{w_{m-2}} < \ldots < \frac{v_{m/2}}{w_{m/2}} < 1,
  \]
  \[
  1 < \frac{v_{2m-m/2+1}}{w_{2m-m/2+1}} < \ldots < \frac{v_{m+2}}{w_{m+2}} < \frac{v_{m+1}}{w_{m+1}}
  \]

where $m/2$ is rounded up if $m$ is odd.
Properties:

- Both ranking schemes of $B_{2m}$ agree with ranking via score vector
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- Among all tournaments with an even number of teams, the Brualdi-Li Matrix has minimal variation in rankings (well-matched teams).
Properties:

- Both ranking schemes of $B_{2m}$ agree with ranking via score vector.
- Among all tournaments with an even number of teams, the Brualdi-Li Matrix has minimal variation in rankings (well-matched teams).
- Left Perron vector and right Perron vector are transposes of each other.
Introducing...

**Types**

- Perron-Frobenius

**Ranking**

- Big Example

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### Probabilities

Let $v$ be the Perron vector of a tournament matrix $A$

- Probability team $i$ beats team $j$ is $\pi_{ij} = \frac{v_i}{v_i + v_j}$

- Generalized tournament matrix $G$: $[G]_{ij} = \pi_{ij}$
Big Example

Consider $B_{12}$:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\end{bmatrix}
\]
The right Perron vector is

\[ \mathbf{v} = \lim_{k \to \infty} \frac{\mathbf{B}_k^{12} \mathbf{1}_{12}}{\| \mathbf{B}_k^{12} \mathbf{1}_{12} \|} = 
\begin{bmatrix}
\end{bmatrix} \]

and the left Perron vector is

\[ \mathbf{w} = \lim_{k \to \infty} \frac{\mathbf{1}_{12}^T \mathbf{B}_k^{12}}{\| \mathbf{1}_{12}^T \mathbf{B}_k^{12} \|} = 
\begin{bmatrix}
\end{bmatrix} \]

with decimals rounded to three significant figures.
Strength to weakness ratios

\[ \frac{v_6}{w_6} = .845 < \frac{v_1}{w_1} = .873 < \frac{v_5}{w_5} = .876 < \frac{v_2}{w_2} = .890 < \frac{v_4}{w_4} = .891 < \frac{v_3}{w_3} = .896 < 1 \]

\[ 1 < \frac{v_{10}}{w_{10}} = 1.116 < \frac{v_9}{w_9} = 1.122 < \frac{v_{11}}{w_{11}} = 1.123 < \frac{v_8}{w_8} = 1.142 < \frac{v_{12}}{w_{12}} = 1.145 < \frac{v_7}{w_7} = 1.184 \]
Ranking according to Kendall-Wei:

12, 11, 10, 9, 8, 7, 1, 2, 3, 4, 5, 6

Ranking according to Ramanucharyula:

7, 12, 8, 11, 9, 10, 3, 4, 2, 5, 1, 6.
Generalized Tournament Matrix:

\[
\begin{bmatrix}
0 & .503 & .506 & .512 & .519 & .530 & .488 & .486 & .483 & .479 & .474 & .466 \\
.494 & .496 & 0 & .506 & .513 & .524 & .482 & .480 & .477 & .473 & .468 & .460 \\
.481 & .483 & .487 & .493 & 0 & .511 & .469 & .467 & .464 & .460 & .455 & .447 \\
.512 & .515 & .518 & .524 & .531 & .542 & 0 & .498 & .495 & .491 & .486 & .478 \\
.514 & .516 & .520 & .526 & .533 & .544 & .502 & 0 & .497 & .493 & .488 & .480 \\
.517 & .520 & .523 & .529 & .536 & .547 & .505 & .503 & 0 & .496 & .491 & .483 \\
.526 & .529 & .532 & .538 & .545 & .556 & .514 & .512 & .509 & .505 & 0 & .492 \\
.534 & .537 & .540 & .546 & .553 & .564 & .522 & .520 & .517 & .513 & .508 & 0
\end{bmatrix}
\]