1. Compute the determinant of $A$ (without using Sage). (15 points)

$$A = \begin{bmatrix} 2 & -1 & 4 & -1 \\ -2 & 2 & 0 & 2 \\ 4 & 3 & 0 & 3 \\ 0 & -1 & -3 & -1 \end{bmatrix}$$

2. Given the matrix $B$ below, find an invertible matrix $S$ such that $S^{-1}BS$ is a diagonal matrix. You may use Sage to compute eigenvalues and to row-reduce matrices. (15 points)

$$B = \begin{bmatrix} -4 & 12 & -18 \\ -6 & 23 & -36 \\ -3 & 12 & -19 \end{bmatrix}$$
3. Consider the square matrix $A$. (40 points)

\[
A = \begin{bmatrix}
-27 & 48 & 31 & -20 \\
-25 & 47 & 25 & -13 \\
40 & -80 & -36 & 13 \\
42 & -84 & -42 & 18
\end{bmatrix}
\]

(a) Use Sage to compute a factored version of the characteristic polynomial.

(b) Without using Sage, determine the eigenvalues of $A$ and their geometric multiplicities.

(c) Using only Sage's reduced row-echelon form method for computational assistance, determine all the eigenspaces of $A$ along with their geometric multiplicities.

(d) Is $A$ diagonalizable? Explain fully.
4. Suppose that $A$ is an $n \times n$ matrix and $\lambda \in \mathbb{C}$. Define

$$U = \{ x \in \mathbb{C}^n \mid A x = \lambda x \}$$

Prove additive closure for $U$, which is one of the three conditions of checking that a set is a subspace. (15 points)

5. Suppose that $A$, $B$ and $C$ are three $n \times n$ matrices that are equal to each other, except for the entries in column $k$. In $A$ column $k$ is the vector $x$, in $B$ column $k$ is the vector $y$, and in $C$ column $k$ is the vector $x + y$. Prove that $\det(C) = \det(A) + \det(B)$. (15 points)

Pictorially: $A = [\ldots |x| \ldots]$ $B = [\ldots |y| \ldots]$ $C = [\ldots |x+y| \ldots]$