Show all of your work and explain your answers fully. There is a total of 100 possible points. You may use Sage to create and row-reduce matrices.

1. Determine if the set of column vectors, \( T \), is linearly independent or not, including an accurate justification for your answer. (15 points)

\[
T = \left\{ \begin{bmatrix} -1 \\ -3 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 4 \\ 3 \\ -2 \end{bmatrix} \right\}
\]

According to Theorem LIVRN we can start by making a matrix with these vectors as columns and "row-reducing"

\[
A = \begin{bmatrix} -3 & 4 & 7 \\ -2 & 1 & 4 \\ -1 & 2 & 3 \\ -1 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

Now we see that \( r = 3 = n \), so by Theorem LIVRN, \( T \) is a linearly independent set.

2. Determine if the vector \( w \) is in the set \( U = \langle T \rangle \). (15 points)

\[
w = \begin{bmatrix} 2 \\ -2 \\ -4 \\ -1 \end{bmatrix} \quad T = \left\{ \begin{bmatrix} 1 \\ -4 \\ 2 \\ 0 \\ 2 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -4 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 2 \\ -2 \end{bmatrix} \right\}
\]

By the definition of a span, we want to know if there are scalars \( a_1, a_2, a_3 \) so that

\[
a_1 \begin{bmatrix} 1 \\ -4 \\ 2 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ -4 \\ -4 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ -5 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}
\]

By SSLE, \( a_1, a_2, a_3 \) are a solution to a system with augmented matrix:

\[
\begin{bmatrix} 1 & -1 & 2 & 2 \\ -4 & 5 & -7 & -2 \\ 2 & -4 & 3 & 4 \\ 0 & 1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

Last column is a pivot column so by Theorem RCLS, the system has no solution. Hence, \( y \notin \langle T \rangle \).
3. Find a linearly independent set \( R \) whose span is the null space of \( A \) (in other words, \( \langle R \rangle = \mathcal{N}(A) \)). Explain how you know your answer has the required properties. (20 points)

\[
A = \begin{bmatrix}
-3 & 1 & -5 & 5 & 3 \\
-4 & 1 & -7 & 6 & 5 \\
-4 & 1 & -7 & 6 & 5
\end{bmatrix}
\]

This is a straightforward application of Theorem BNS. We need a row-reduced version of \( A \).

\[
A \xrightarrow{\text{rref}} \begin{bmatrix}
1 & 0 & 2 & -1 & -2 \\
0 & 1 & 1 & 2 & -3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis: \( F = \{3, 4, 5\} \)

\[
R = \left\{ \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix} \right\} = \left\{ \begin{bmatrix}
-2 \\
-1 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
1 \\
\end{bmatrix}, \begin{bmatrix}
2 \\
3 \\
0 \\
\end{bmatrix} \right\}
\]

\( \rightarrow \) slots 3, 4, 5

owing to free variables \( x_3, x_4, x_5 \)

in homogeneous system \( LS(A, \mathbf{0}) \)

Theorem BNS gives span \& linear independence.

4. Find a linearly independent set \( T \) with the same span as \( S \) (in other words \( \langle T \rangle = \langle S \rangle \)). (20 points)

\[
S = \left\{ \begin{bmatrix}
1 \\
1 \\
0 \\
-1
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
1 \\
-1
\end{bmatrix}, \begin{bmatrix}
-2 \\
-5 \\
-3 \\
-1
\end{bmatrix}, \begin{bmatrix}
-3 \\
1 \\
5 \\
0
\end{bmatrix}, \begin{bmatrix}
4 \\
0 \\
0 \\
0
\end{bmatrix} \right\}
\]

Apply Theorem BNS.

Create a matrix whose columns are the vectors of \( S \), and row-reduce.

\[
B = \begin{bmatrix}
1 & -1 & -2 & -3 & 4 \\
1 & 0 & -5 & 1 & 0 \\
0 & 1 & -3 & 5 & 5 \\
-1 & -1 & 8 & -1 & 0
\end{bmatrix}
\xrightarrow{\text{rref}} \begin{bmatrix}
1 & 0 & -5 & 0 & 1 \\
0 & 1 & -3 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis:

\( D = \{1, 2, 3, 4\} \)

\( T \) is the columns numbered 1, 2 \& 4 from \( B \):

\[
T = \left\{ \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
-3 \\
0 \\
1
\end{bmatrix} \right\}
\]
5. Suppose that \( u \in \mathbb{C}^n \) is a vector. Prove that \( 1u = u \). Provide reasons for each deduction and employ our indexing notation for entries of vectors. (Do not simply quote this as a result from Theorem VSPCV.) (15 points)

For \( 1 \leq i \leq n \), scalar equality

\[
[1u]_i = 1 [u]_i \quad \text{Definition CVSM}
\]

\[
= [u]_i \quad \text{Property OCN}
\]

So by Definition CVE, \( 1u = u \).

6. Suppose that \( A = [A_1 | A_2 | A_3 | \ldots | A_n] \) is a matrix and that both \( x, y \in \mathbb{C}^n \) are solutions to \( LS(A, b) \). Prove that \( x - y \) is a solution to the homogeneous system \( LS(A, 0) \). (You may assume that vector subtraction is defined by \([x - y]_i = [x]_i - [y]_i \); for \( 1 \leq i \leq n \).) (15 points)

Consider \([x - y]_1 A_1 + [x - y]_2 A_2 + \ldots + [x - y]_n A_n\)

\[
= ([x]_1 - [y]_1) A_1 + ([x]_2 - [y]_2) A_2 + \ldots + ([x]_n - [y]_n) A_n
\]

\[
= [x]_1 A_1 - [y]_1 A_1 + [x]_2 A_2 - [y]_2 A_2 + \ldots + [x]_n A_n - [y]_n A_n
\]

\[
= ([x]_1 A_1 + [x]_2 A_2 + \ldots + [x]_n A_n) - ([y]_1 A_1 + [y]_2 A_2 + \ldots + [y]_n A_n)
\]

\[
= b - b = 0
\]

By SLSLC this says \( x - y \) is a solution to \( LS(A, 0) \).