A Combinatorial Analysis of Finite Boolean Algebras

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May 1, 2013

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1 Introduction

Boolean algebras serve an important purpose in the study of algebraic systems, providing algebraic structure to the notions of order, inequality, and inclusion. The algebraist is always trying to understand some structured set using symbol manipulation. Boolean algebras are then used to study the relationships that hold between such algebraic structures while still using basic techniques of symbol manipulation. In this paper we will take a step back from the standard algebraic practices, and analyze these fascinating algebraic structures from a different point of view. Using combinatorial tools, we will provide an in-depth analysis of the structure of finite Boolean algebras.

We will start by introducing several ways of analyzing poset substructure from a combinatorial point of view. After proving several results about posets, one of which is highly nontrivial, we will then see how we can apply these tools to the study of Boolean algebras. We will build towards a central result known as "Sperner's theorem," before demonstrating several ways of extending the ideas behind this seminal theorem.

This paper intends to introduce students that are well versed in basic abstract algebra to algebraic and extremal combinatorics. There is a vast interplay between algebra and combinatorics at all levels of both theories, making knowledge of combinatorial theory highly valuable to the student of algebra, and vice-versa. It should be noted that this article is intended to be an extension of the material in [2], and as such, results that have been proven in [2] will often be assumed here, although a reference to their proof will be given.

2 Basic Concepts

Though the ultimate goal of this paper will be to analyze Boolean algebras, the tools we will be using are more generally applicable. Many of these definitions will be stated in terms of posets. A Boolean algebra is nothing more than a poset with added structure, so all definitions and propositions about posets can be extended to describe Boolean algebras. We will present these concepts in their full generality, as they are traditionally defined.

Recall from chapter 19 of [2] that a *poset* is a set, X, together with a relation, $\preceq$, such that:

1. $\preceq$ is reflexive, i.e. $\forall x \in X \; x \preceq x$
2. $\preceq$ is antisymmetric, i.e. $\forall x, y \in X \; x \preceq y \;\&\; y \preceq x$ implies $y = x$
3. $\preceq$ is transitive, i.e. $\forall x, y, z \in X \; x \preceq y \;\&\; y \preceq z$ implies $x \preceq z$

While this definition allows for posets of any cardinality, in this paper we will only be concerned with finite posets. Thus, when we use the term *poset* below, we will always be speaking of a finite poset.

2.1 Chains

Any comprehensive treatment of a mathematical structure will surely include an extensive discussion of its various substructures. We will start by a particularly important poset substructure, which, together with its dual, provides the foundation for many compelling combinatorial problems.

The definitions in this section are derived roughly from those provided in [4].

Definition 2.1: Let P be a poset. A subset $C \subseteq P$ is called a *chain* if for all $x, y \in C$, either $x \preceq y$ or $y \preceq x$, where C is endowed with the same order relation as P.

For the sake of brevity, we will say that two elements of a poset, x and y, are *comparable* if $x \preceq y$ or $y \preceq x$. Note that a chain is nothing more than a totally ordered subset of P. We will say that a chain has *length* n if it contains $n + 1$ elements.

Example 2.2: Let P be the set of the divisors of 30 ordered by divisibility. A Hasse diagram for this poset is shown in Figure 2.1. The set $C_1 = \{30, 15, 5, 1\}$ is a chain of length 3, as $1 \preceq 5 \preceq 15 \preceq 30$, while $C_2 = \{6, 1\}$ is a chain of length 1. However, the set $\{2, 3, 6\}$ is not a chain, as 2 and 3 are not comparable. It should be easy to verify that one can find a chain by iteratively tracing lines of the diagram connecting some element to another that is vertically below it.

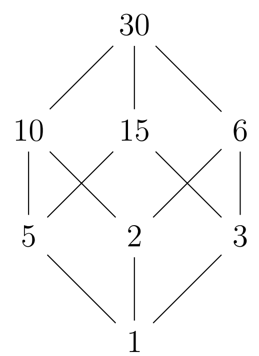

Figure 2.1: Poset of divisors of 30 ordered by divisibility

Given a poset P, we say that x *covers* y in P if $y \prec x$ and there is no z such that $y \prec z \prec x$. Thus, looking back to Figure 2.1, we can say that 10 covers 5, but 30 does not cover 2, even though $2 \prec 30$. A *maximal* chain is a chain that is contained in no longer chain. Again referring back to Figure 2.1, we can then see that C_1 is maximal, while C_2 is not, as it is contained within the chain $\{30, 6, 1\}$.

We will say that a chain $x_1 \prec x_2 \prec ... \prec x_n$ is *saturated* if x_{i+1} covers x_i for all indices i such that $1 \leq i \leq n-1$.The following proposition will allow us to fully characterize maximal chains.

Proposition 2.3: Let P be a poset and $C = \{x_1 \prec x_2 \prec ... \prec x_n\}$ be a chain in P. Then C is maximal if and only if C is saturated and we can find no elements in P either larger than x_n or smaller than x_1.

Proof: First, assume that C is saturated and that there are no elements in P larger than x_n or smaller than x_1. Let y be some element that is comparable to all of the elements of C. The minimality and maximality of x_1 and x_n, respectively, means that $x_1 \preceq y \preceq x_n$.

Let x_i be the smallest element of C such that $y \preceq x_i$. The minimality of x_i implies that $y \npreceq x_{i-1}$, so, because y is comparable to every element in C, $x_{i-1} \prec y$. Because x_i covers x_{i-1}, it cannot be the case that $y \prec x_i$. Thus, it must be the case that $y = x_i$. Therefore, every element in P that is comparable to all elements of C is in C, so there is no element that can be added to C to create a larger chain.

We will prove the converse by demonstrating its contrapositive. Suppose that the statement "C is saturated and we can find no elements in P either larger than x_n or smaller than x_1" is false. Then there are three possible cases. (1) There is some $z \in P$ such that $x_n \prec z$, meaning that C is properly contained within the chain $x_1 \prec x_2 \prec ... \prec x_n \prec z$. (2) There is some $w \in P$ such that $w \prec x_1$, meaning that C is properly contained within the chain $w \prec x_1 \prec ... \prec x_n$. Or, (3) There is some $x_{i+1} \in C$ such that x_{i+1} does not cover x_i, meaning that there is some element $u \in P$ such that $x_i \prec u \prec x_{i+1}$, so that C is properly contained in the chain $x_1 \prec ... \prec x_i \prec u \prec x_{i+1} ... \prec x_n$. In all cases C is not maximal.

■

We have a few more basic definitions to present before moving on to our first substantial result.

Definition 2.4: Let P be a poset. We say that P is *graded of rank n* if all maximal chains in P are of length n.

If x is an element of a graded poset, P, of rank n, we say that x has *rank j*, $\rho(x) = j$, if the longest saturated chain in P with x as a top element is of length j. Note that if we let $P_j = \{x \in P | \rho(x) = j\}$, a set which we will call the jth level of P, then P_i, $0 \leq i \leq n$, is a partition of P. We will denote $|P_j|$ as p_j.

Example 2.5: Let P be the diamond poset of size 6 illustrated in Figure 2.2(a). We can see that every maximal chain in P is of the form $0 \prec x \prec 1$, where $x \in \{a, b, c, d\}$. Thus, P is graded of rank 2. Also note that $a, b, c,$ and d all have rank 1, while 1 has rank 2. Finally, we can see that P is partitioned into $P_0 = \{0\}$, $P_1 = \{a, b, c, d\}$ and $P_2 = \{1\}$.

The poset illustrated in Figure 2.2(b), however, is not graded of rank n, as $\{a, b\}$ is a maximal chain of length 1, while $\{a, c, d\}$ is a maximal chain of length 2.

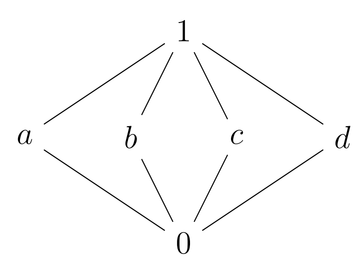

Figure 2.2(a): Diamond poset of size 6

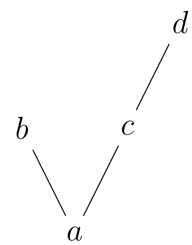

Figure 2.2(b): A poset that is not graded

2.2 Antichains

Having discussed briefly the structure general notion of a chain, we now turn to its dual structure, which will be the central object of study in most of our later results.

Definition 2.6: Let P be a poset. A subset $A \subset P$ is an *antichain* if $\forall x, y \in A$, x and y are **not** comparable.

The *width* of a poset is the size of its largest antichain, denoted $w(P)$. Looking to the examples above, we see that in the diamond poset of size 6 (Figure 2.2(a)), any subset of $\{a, b, c, d\}$ is an antichain, while the largest antichain is $\{a, b, c, d\}$ itself, meaning that this poset has width four. Looking now to the poset shown in Figure 2.2(b), we see that both $\{b, c\}$ and $\{b, d\}$ are antichains, and they are both of maximum size, so the width of this poset is 2. Also note that both 0 and 1 in Figure 2.2(a) cannot be part of any nontrivial antichain (i.e. an antichain of size greater than 1) as they are both comparable to every other element. Finally, if we consider the set of integers between 1 and 10 ordered by the traditional less than relation, we see that we can have no nontrivial antichains, so the width of this poset is 1. [4]

3 Dilworth's Chain Decomposition Theorem

Having defined and described the various combinatorial tools we will be using, we are now nearly ready to apply them to the study of Boolean algebras. However, one of the results we will prove about Boolean algebras is easily generalizable, and to state this result with its full power, we will prove it now, before stepping out of our generalized environment. We start by defining two important families of subposets that should seem familiar, as they are analogues of their ring-theoretic namesakes.

Definition 3.1: Let P be a poset. A subset $I \subseteq P$ is called an *ideal* if $x \in I$ and $y \preceq x$ imply $y \in I$. Meanwhile, a subset $R \subseteq P$ is called a *dual ideal* if $x \in I$ and $x \preceq y$ imply that $y \in I$.

Just as a principal ideal in ring theory is of the form $\langle a \rangle = \{ar | r \in R\}$, we will call a poset ideal principal if it is of the form $I(a) = \{x \in P | x \preceq a\}$, with a similar construct for dual ideals. Thus, we can see that $I(d) = \{d, c, a\}$ is an ideal in the poset shown in Figure 2.2(b), while $D(15) = \{15, 5, 3, 1\}$ is a dual ideal in the poset shown in Figure 2.1.

We will now prove a quick lemma that motivates our larger theorem.

Lemma 3.2: Let P be a poset. If we can partition P into m chains, then $w(P) \leq m$.

Proof: If we can partition P into m chains, then we cannot have an antichain of size greater than m, as an antichain can intersect each of these chains at most once before it contains two comparable elements. Because we know we can form an antichain of size $w(P)$, any partition we have of P into chains must contain at least $w(P)$ chains.

■

The preceding lemma is fairly trivial, but its inclusion is necessary to provide background for the following theorem. Where the lemma says the size of a partition of a poset into chains must be at least as large as the width of the poset, the theorem asserts that we can partition a poset into exactly $w(P)$ chains. It was introduced and first proven by Dilworth in 1950, but the proof we will now present is based on the exposition of Perles' 1963 proof presented in [3].

Theorem 3.3: Let P be a poset of width k. Then P can be partitioned into k chains.

Proof: We will proceed by induction on the size of P. If $|P| = 1$, so that $P = \{x\}$, then there is only one antichain, namely $\{x\}$, which is of size 1, so $w(P) = 1$. Also, P can be partitioned into one chain, namely $C = \{x\}$.

Now, we will assume that we have a poset P of size n, and for all posets of size m, $1 \leq m \leq n$, the given statement holds.

If the only maximal antichain in $A \subseteq P$ is equal either to the set of maximal elements of P or to the set of minimal elements of P, then we can pick out a maximal element a and a minimal element b such that $b \preceq a$. Then $P \setminus \{a, b\}$ must have width $w(P) - 1$, as one of a or b is in its only maximally sized antichain. Because $P \setminus \{a, b\}$ is of size $n - 2$, we can apply the induction hypothesis to see that $P \setminus \{a, b\}$ has a partition into $w(P) - 1$ chains. Then, if we add the chain $C = \{a, b\}$ to this partition, we have a partition of P into $w(P)$ chains.

We are then left with the case that there is some antichain $A \subseteq P$ that is not equal to the set of maximal elements nor the set of minimal elements. In this case, we consider the ideal $I(A) = \bigcup_{a \in A} I(a) = \{x \in P | \exists a \in A \text{ s.t. } x \preceq a\}$ and the dual ideal $D(A) = \bigcup_{a \in A} D(a) = \{x \in P | \exists a \in A \text{ s.t. } a \preceq x\}$. Because A is maximal, every element in P is comparable to some element of A, so $I(A) \cup D(A) = P$, while the antisymmetry of the order relation implies $I(A) \cap D(A) = A$. Meanwhile, because A is neither the set of minimal elements nor the set of maximal elements, we know that $|I(A)| < n$ and $|D(A)| < n$, and can apply the induction hypothesis to see that both $I(A)$ and $D(A)$ have partitions into $w(P)$ chains, as the fact that they both contain A implies they both have width $w(P)$. The largest element of every chain in $I(A)$ is some unique element of A, while the smallest element of every chain in $D(A)$ is some unique element of A. Thus, we can "splice" the chains that have intersecting maximal/minimal elements to get a partition of P into $w(P)$ chains.

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4 Boolean Algebras

We are now set to "reintroduce" readers to Boolean algebras, and discuss them in a combinatorial setting. One should recall from [2] that a *Boolean algebra* is a set B with two distinguished elements, usually denoted I and O, and three operations defined on its elements (the binary operations $a \wedge b$ and $a \vee b$, and the unary operation a'). Recall also that it is stipulated that these binary operations be associative, symmetric, and distribute across each other, and that $a \wedge O = O$ and $a \vee I = I$ for all a in B. For a full definition of these structures, one should look to pages 311-312 of [2].

If B is a Boolean algebra, we can define an order on B such that $a \preceq b$ if and only if $a \wedge b = a$. We refer the reader to page 313 of [2] to see the proof that B is indeed a poset. Also note that an *atom* $a \in B$ is defined to be an element such that there exists no $b \in B$ "in between" O and a, i.e. there is no b satisfying $O \prec b \prec a$. As was stated above, in this paper we are discussing only finite sets. Theorem 19.12 of [2] demonstrated that every Boolean algebra is isomorphic to the power set of the set of its atoms. In the discipline of algebra, we are only concerned with describing various structures *up to isomorphism*. Thus, we will henceforth use the set $[n] = \{1, 2, ..., n\}$ to represent the atoms of a given Boolean algebra, and $P([n])$ will then be *the* Boolean algebra of order 2^n, B_n. We are assuming that the reader is aware of the fact that the power set of a set of size n has size 2^n, which can be seen by realizing that the size of the power set is just the sum of the number of subsets that can be formed of each possible size, and $\sum_{i=1}^{n} \binom{n}{i} = (1+1)^n = 2^n$. We will then equate $\vee$ with $\cup$ (the union operation), $\wedge$ with $\cap$ (the intersection operation), $\preceq$ with $\subseteq$ (the subset relation), and the prime operation with that of taking complements.

We turn now to our combinatorial analysis of these algebraic structures.

Proposition 4.1: B_n, the Boolean algebra of order 2^n, is graded of rank n.

Proof: We are trying to show that every maximal chain in B_n is of length n. Because every maximal chain must contain both an element that is smaller than no other in B_n and larger than no other in B_n, we see that any maximal chain must contain both $\varnothing$ and $[n]$. We also know that a maximal chain must be saturated. It should be clear that for two sets $R \subset S$, there is no set that can be inserted between them if and only if there is a single element x of the universal set such that $S = R \cup \{x\}$. Thus, we can form a bijection between the members of the base set and the coverings in the chain, where (using the notation from the previous sentence) S covering R would be mapped to x. That this function is injective is obvious, and that it is onto can be seen by realizing that this process of adding a single element must traverse B_n from $\varnothing$ to $[n]$. Therefore, a maximal chain must be of length n.

■

Example 4.2: Refer back to Figure 2.1. It should be easy to see that if we map each element of this poset to its set of prime divisors, this structure is then isomorphic to B_3. Because 1 is the only minimal element, and 30 is the only maximal element, every maximal chain most contain both 1 and 30. Also, because every maximal chain must be saturated, it must add no more than one prime divisor each time a new element is added to the chain. Thus, it should be clear that this Boolean algebra is graded of rank 3. Note, however, that not every graded poset is a Boolean algebra, as the poset shown in Figure 2.2(a) is graded of rank 2, but clearly cannot be isomorphic to any Boolean algebra because it contains 6 elements, whereas the order of every Boolean algebra is a power of 2.

That a Boolean algebra is graded of rank n should come as no surprise, as its possession of maximal and minimal elements almost guarantees this. This next result, however, should not be so apparent prior to its demonstration, although its proof will be particularly succinct and understandable. To understand this proposition, one must recall that a *lattice* is a slight generalization of a Boolean algebra (meaning that every Boolean algebra is a lattice), where it is not required that $\wedge$ and $\vee$ distribute properly, the unary operator does not exist, and maximal and minimal elements I and O need not be members of the set. Because a subset of a lattice, when endowed with the same operations as its superset, inherits important structural aspects, the only condition necessary to determine if a given set is a sublattice is that it is closed under $\vee$ and $\wedge$.

Proposition 4.3: Let B_n be the Boolean algebra of size 2^n. Then the largest sublattice of B_n not containing the empy set is of size 2^{n-1}.

Proof: Because a sublattice L of B_n must be closed under intersections, it cannot contain $x, y \in B_n$ such that $x \cap y = \varnothing$. Letting x^c denote the complement of x, we realize that, by the definition of set complements, $x \cap x^c = \varnothing$. Thus, for every $x \in L$, $x^c \notin L$, and because every set has a unique complement, only half of the elements of B_n can be in L, so we have our upper bound. Now, remembering that $B_n = P([n])$, let $L = \{x \in B_n | 1 \in x\}$. If $x, y \in B_n$ both contain 1, clearly so do $x \cap y$ and $x \cup y$. Each set containing 1 can be uniquely paired with a set not containing 1, namely its complement, so $|L| = 2^{n-1}$, and we have our desired lower bound.

■

Note that the example given for the lower bound in the previous proof can be easily mapped to $P([n] \setminus \{1\})$ by removing 1 from each subset, and therefore is itself a Boolean algebra isomorphic to B_{n-1}.

5 Sperner's Theorem

We are now set to prove our first major structural theorem concerning Boolean algebras, from which a second valuable result will follow as a corollary. The theorems proven in this section and the next will provide the "meat" of our combinatorial analysis of Boolean algebras.

5.1 The Sperner Property

We said above that we were moving on from the generality of posets to a specific discussion of Boolean algebras, but we must now take a short step back in order to provide a full picture of this paper's central theorem. Given a poset, it is natural to ask what sort of bounds we can place on the size of its largest antichain. In 1927, Emanuel Sperner was able to find a strict value for the size of the largest antichain in a Boolean algebra [4]. Combinatorial mathematicians have since then been trying to answer similar questions for other types of posets. Certain posets have been shown to have the same bound as Boolean algebras, so a name has been given to this property.

Definition 5.1: Let P be a graded poset of rank n. We say that P has the *Sperner property* if $\max\{|A|\,|A \text{ is an antichain of } P\} = \max\{P_j|1 \leq j \leq n\}$, i.e. the size of the largest antichain in P is the size of the largest rank level P_j.

If a poset has the Sperner property, we call it a *Sperner poset.* Note that if a poset has the Sperner property it may still have maximally sized antichains besides its largest rank level P_j [4].

Example 5.2: The poset in Figure 5.1(a) has the Sperner property. It should be relatively easy to see that this poset is graded of rank 2. Meanwhile, any attempt to construct an antichain of size 4 fails. Note that this set contains a maximally sized antichain that is not a rank level, namely $\{c, d, b\}$. The poset in Figure 5.1(b), on the other hand, is not a Sperner poset. Although it is clearly graded of rank 1, as it has no chains of length longer than 1, the set $\{a, c, e, f\}$ forms an antichain of size 4 even though each of its rank levels are only of size 3.

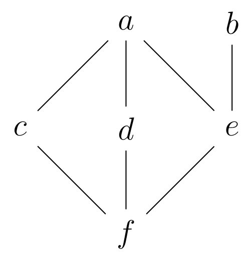

Figure 5.1(a) A poset with the Sperner property

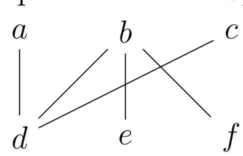

Figure 5.1(b): A graded poset without the Sperner property

5.2 Sperner's Theorem

Sperner's theorem states that all Boolean algebras have the Sperner property. The proof we provide here employs a particularly nice lemma, proven independently by Lubell (in 1966), Yamamoto (in 1954), and Meschalkin (in 1963). It was been aptly named the *LYM inequality.* We should stress that, although this inequality will serve as no more than a lemma in the present paper, it is actually a rather powerful tool in extremal set theory. Our proof will follow that given by Anderson in [1].

Lemma 5.3: Let B_n be a Boolean algebra of size 2^n, and let A be an antichain in B_n. If we let q_k denote the number of members of A in the rank level P_k, then

$$\sum_k \frac{q_k}{\binom{n}{k}} \leq 1$$

Proof: We will start by noting that there are $n!$ permutations of $[n]$, a fact given on page 77 of [2]. We will say that a permutation π "begins with" a set x if π sends the points $1, 2, ..., |x|$ to the members of x in some order. There are $|x|!(n-|x|)!$ such permutations, as we have $|x|$ places to which we can map the first $|x|$ elements, and $n-|x|$ places to map the rest. Also, because A is an antichain, if $x, y \in A$, then no permutations can start with both x and y, as then one would be a subset of the other. This means that the set of permutations beginning with one member of A is disjoint from the set of permutations beginning with any other member of A, so

$$\sum_k k!(n-k)!q_k = \sum_{x \in A} |x|!(n-|x|)! \leq n!$$

from which we get the desired result by dividing both sides by $n!$.

■

This result seems, in and of itself, fairly unmotivated, and perhaps even unimportant. However, we will see the beauty and power of this inequality shortly. This proof will once again be following [1].

Theorem 5.4: B_n has the Sperner property.

Proof: We start by noting that each rank level, P_i, in B_n is just the the number of sets of size i containing elements of $[n]$. So, for all i, $p_i = \binom{n}{i}$, which is maximized when $i = \lfloor \frac{n}{2} \rfloor$, the largest integer smaller than $\frac{n}{2}$. We consider this a basic combinatorial fact, and refer a confused reader to page 2 of [1]. Now, let A be an antichain in B_n. Using the same notation q_k as was used in the preceding lemma

$$|A| = \sum_k q_k = \binom{n}{\lfloor \frac{n}{2} \rfloor} \sum_k \frac{q_k}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \sum_k \frac{q_k}{\binom{n}{k}} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \tag{1}$$

Where the last step employs Lemma 5.3. We complete the proof by noting that B_n has an antichain of size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, namely $P_{\lfloor \frac{n}{2} \rfloor}$.

■

We can now say that $w(B_n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Thus, we are able to look back to Dilworth's chain decomposition theorem, the subject of Section 3, and obtain a further result about the structure of Boolean algebras.

Corollary 5.5: The Boolean algebra of size 2^n, B_n, can be partitioned into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ disjoint chains.

Proof: This results follows directly from Theorem 3.3 and Theorem 5.4.

$\blacksquare$

Example 5.6: Recall from Example 4.2 that the poset shown in Figure 2.1 is the Boolean algebra B_3 with a different labeling of the vertices. Corollary 5.5 then tells us that we can partition this poset into $\binom{3}{\lfloor \frac{3}{2} \rfloor} = 3$ disjoint chains. Refer to Figure 5.2 to see how this is possible.

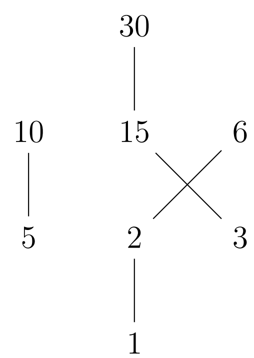

Figure 5.2: A partition of B_3 into 3 disjoint chains

6 Extensions

The importance of Sperner's theorem in the combinatorial study of Boolean algebras should not be understated. It has inspired countless mathematicians to both ask further questions about the chain/antichain structure of Boolean algebras, and to employ similar techniques in combinatorially analyzing other finite sets. We will conclude the mathematical work of this paper by proving two theorems that provide further insight into the structure of Boolean algebras while demonstrating typical problems and techniques of Sperner theory.

6.1 Maximally Sized Antichains

Having seen Sperner's theorem, it is natural to ask if there are any antichains of maximal size besides the largest rank levels. That there are no such chains was proven by Lovasz in 1979. Our proof will follow the presentation of Lovasz's proof in [1].

Theorem 6.1: The only antichains of maximal size in B_n are rank levels. Namely, when n is even, the only antichain of maximal size is $P_{\frac{n}{2}}$, while for odd n $P_{\frac{n+1}{2}}$ and $P_{\frac{n-1}{2}}$ are the only two antichains of maximal size.

Proof: Looking back at (1), we see that equality in this equation is achieved only when $\binom{n}{k} = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for all k. Therefore, when n is even we can only have sets of size $\lfloor \frac{n}{2} \rfloor$ in a maximal antichain. However, if n is odd, then we have only ruled out elements outside of $P_{\frac{n+1}{2}} \cup P_{\frac{n-1}{2}}$, and have not ruled out some antichain that combines elements of these two rank levels.

We will prove this cannot be the case by contradiction. For convenience's sake, let $n = 2m + 1$. Suppose that A is an antichain containing some, but not all, of the sets in

P_{m+1}. Then, we can choose $X, Y \in P_{m+1}$ such that $X \in A$, but $Y \notin A$. We can then label the elements of $[n]$ so that $X = \{n_1, n_2, ..., n_{m+1}\}$ and $Y = \{n_i, n_{i+1}, ..., n_{i+m}\}$, for some index i. Because $X \in A$ while $Y \notin A$, there is then some smallest index j, where $1 \leq j < i$, such that $Z = \{n_j, n_{j+1}, ..., n_{j+m}\} \in A$ while $W = \{n_{j+1}, n_{j+2}, ..., n_{j+m+1}\} \notin A$. The fact that $Z \cap W \subseteq Z$ tells us that $Z \cap W \notin A$, because A is an antichain containing Z.

However, there is a further condition for equality in (1) that we have yet to discuss, namely that the LYM inequality be a statement of equality. Our very first step in deriving that inequality was to note that there is a unique set of permutations beginning with each element of A, so that if we sum up the number of permutations beginning with x, for each $x \in A$, our total must be at most the number of permutations of $[n]$. From this fact, we used nothing more than algebraic manipulation to derive the ultimate inequality, so we see that there must be equality in the LYM inequality if and only if every permutation of $[n]$ begins with some element of A.

We now look back at $Z \cap W = \{n_{j+1}, n_{j+2}, ..., n_{j+m}\}$. By the last line of the previous paragraph, the permutation that begins with $n_{j+1}, n_{j+2}, ..., n_{j+m+1}$ must be a permutation that begins with some element of A. If we remember that in the first paragraph of this proof we ruled out the possibility that a member of A can be any size but m or $m+1$, then we see that either W or $Z \cap W$ must be in A, for these are the only two sets of the correct size that contain the elements $n_{j+1}, n_{j+2}, ..., n_{j+m+1}$. In either case we have a contradiction, as we previously stipulated that each of these sets are not members of A. Thus, we can conclude that, if n is odd, there is no antichain in B_n besides $P_{\frac{n+1}{2}}$ and $P_{\frac{n-1}{2}}$.

■

6.2 The Erdos-Ko-Rado Theorem

As was mentioned above, Sperner's work inspired many other mathematicians to work towards characterizing extremal chains and antichains in finite Boolean algebras. We provide here a famous result in the branch of combinatorics that has come to be called Sperner theory, thereby offering one last look into the combinatorial structure of Boolean algebras. This theorem was first published by Erdos, Ko, and Rado in 1961, but we will provide a more eloquent proof, first given by Katona in 1972. Our presentation will follow that of [5].

Theorem 6.2: Let B_n be the Boolean algebra of size 2^n, and let P_k, $k \leq \frac{n}{2}$, be the kth rank level in B_n. If $C \subseteq P_k$ is a collection of sets such that any two sets in C have a nonempty intersection, then $|C| \leq \binom{n-1}{k-1}$.

Proof: Let $F = \{F_1, F_2, ..., F_n\}$ contain all subsets of P_k of the form $\{i, i+1 \text{ (mod n)}, ..., i+k-1 \text{ (mod n)}\}$, where $1 \leq i \leq n$, and the residue class usually represented by 0 is now represented by n. So, for example, $F_{n-3} = \{n-3, n-2, n-1, n, 1, ..., k-4\}$. From now on we will assume that arithmetic inside members of F is modular. Suppose there is some index j such that $F_j \in C$. We can see that every set in F intersecting F_j is either of the form $\{l, l+1, ..., l+k-1\}$ or of the form $\{l-k, l-k+1, ..., l-1\}$, for some index l such that $j < l \leq j+k-1$. So, there are $2k-2$ sets in F intersecting F_j. We know that only one of F_l or F_{l-k} can be in C while still retaining pairwise nonempty intersections, so only $k-1$ sets in F besides F_j could then also be in C, meaning that $|F \cap C| \leq k$.

Note that our argument in the previous paragraph did not depend on the elements of each F_j being in increasing modular order, but rather they were just presented that way for clarity's sake. Thus, if we let F^π be the set obtained from F when some permutation π is applied to the elements of $[n]$ in each of the F_is, then $|F^\pi \cap C| \leq k$. So,

$$\sum_{\pi \in S_n} |F^\pi \cap C| \leq k \cdot n!$$

We can reformulate the left side of this sum by realizing that each pair of sets (F_i, A), where $A \in C$, is counted exactly once for each permutation that maps F_i to A. If we fix F_i and A, there are $k!(n-k)!$ permutations that take A to F_i, as there are k choices about where to map the elements of F_i, and $n-k$ choices about where to map the rest of the elements of $[n]$. Remembering that $|F| = n$, we see

$$|C| \cdot n \cdot k!(n-k)! \leq k \cdot n!$$

Which can be rearranged to form the inequality given in the theorem's statement.

■

7 Conclusion

Though it is hoped that the reader now understands Boolean algebras to a much greater degree than before picking up this paper, we have barely scratched the surface of Sperner theory, or indeed extremal set theory in general. While Boolean algebras are perhaps the simplest algebra of sets that an extremal set theorist will work with, we have nonetheless left out myriad beautiful and illuminating results concerning their structure for the sake of brevity (this is, of course, not a dissertation, or even a senior thesis). All of the resources listed below would be of value to the interested reader, with [1] in particular focusing entirely on similar material. Extremal set theory is a young yet flourishing branch of discrete mathematics, and is wide open for anyone with some understanding of abstract algebra to attack.

References

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