Math 290
Exam 7
Chapter R

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
As justification for your work, you may use Sage routines designed for matrices, but not routines designed for linear transformations.

1. Consider the following linear transformation from the vector space of polynomials with degree at most 2 to the vector space of 1×2 matrices. (30 points)

$T\colon P_2 \to M_{12}, \quad T\left(a + bx + cx^2\right) = \begin{bmatrix} 2a - 3b + c & a - b + c \end{bmatrix}$

(a) Find the matrix representation relative to the bases $B = \left\{1 + x, 2x + x^2, -4 + 5x + 4x^2\right\}$ and $C = \left\{\begin{bmatrix} 4 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 4 \end{bmatrix}\right\}$.

$$\rho_C(T(1+x)) = \rho_C([-1\ \ 0]) = \rho_C(-4[4\ 5] + 5[3\ 4]) = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

$$\rho_C(T(2x+x^2)) = \rho_C([-5\ \ -1]) = \rho_C(-17[4\ 5] + 21[3\ 4]) = \begin{bmatrix} -17 \\ 21 \end{bmatrix}$$

$$\rho_C(T(-4+5x+4x^2)) = \rho_C([-19\ \ -5]) = \rho_C(-61[4\ 5] + 75[3\ 4]) = \begin{bmatrix} -61 \\ 75 \end{bmatrix}$$

$$M^T_{B,C} = \begin{bmatrix} -4 & -17 & -61 \\ 5 & 21 & 75 \end{bmatrix}$$

(b) Use a version of the Fundamental Theorem of Matrix Representation and your answer to part (a) to compute $T(2 + x + 3x^2)$.

$$\rho_B(2+x+3x^2) = \rho_B(-26(1+x) + 31(2x+x^2) + (-7)(-4+5x+4x^2)) = \begin{bmatrix} -26 \\ 31 \\ -7 \end{bmatrix}$$

$$T(2+x+3x^2) = \rho_C^{-1}\left(M^T_{B,C}\,\rho_B(2+x+3x^2)\right) = \rho_C^{-1}\left(\begin{bmatrix} 4 \\ -4 \end{bmatrix}\right) = 4[4\ 5] + (-4)[3\ 4] = [4\ \ 4]$$

Check: $T(2+x+3x^2) = [2\cdot 2 - 3\cdot 1 + 3 \quad 2 - 1 + 3] = [4\ \ 4]$

(c) Find the kernel of T, $\mathcal{K}(T)$, using your answer in part (a).

First $N(M^T_{B,C})$; $M^T_{B,C} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \end{bmatrix}$

BNS $\Rightarrow$ $N(M^T_{B,C}) = \left\langle\left\{\begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix}\right\}\right\rangle$. Now apply ρ_B^{-1} according to KNSI

$$K(T) = \left\langle\left\{2 + x - x^2\right\}\right\rangle$$

$$\rho_B^{-1}\left(\begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix}\right) = 6(1+x) + -5(2x+x^2) + (-4+5x+4x^2) = 2 + x - x^2$$

Check: $T(2+x-x^2) = [0\ \ 0] = \mathbf{0}$

2. Consider the linear transformation, S. (25 points)

$S\colon M_{22} \to P_3$,

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (9a - 2b - 7c - 5d) + (5a - 4b - 9c - 4d)\,x + (-3a + b + 3c + 2d)\,x^2 + (-2a + c + d)\,x^3$$

(a) Build a matrix representation of S and from this determine *as efficiently as possible* that S is invertible.

Representation relative to standard bases, on sight

$$M^S_{B,C} = \begin{bmatrix} 9 & -2 & -7 & -5 \\ 5 & -4 & -9 & -4 \\ -3 & 1 & 3 & 2 \\ -2 & 0 & 1 & 1 \end{bmatrix}$$

$\det(M^S_{B,C}) = \underline{1} \neq 0$

$\Rightarrow$ matrix representation invertible (SMZD)

$\Rightarrow$ linear transformation invertible

(b) Find an expression for the outputs of the inverse linear transformation S^{-1}. (In other words, find a "formula" or "rule" for the inverse linear transformation.)

$S^{-1} : P_3 \to M_{22}$

$M^{S^{-1}}_{C,B} = (M^S_{B,C})^{-1}$

$$S^{-1}(a+bx+cx^2+d) = \rho_B^{-1}\left(M^{S^{-1}}_{C,B}\,\rho_C(a+bx+cx^2+d)\right)$$

$$= \rho_B^{-1}\left(\begin{bmatrix} 1 & 0 & 2 & 1 \\ -2 & 1 & 1 & -8 \\ 1 & -1 & -2 & 5 \\ 1 & 1 & 6 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \rho_B^{-1}\left(\begin{bmatrix} a+2c+d \\ -2a+b+c-8d \\ a-b-2c+5d \\ a+b+6c-2d \end{bmatrix}\right)$$

$$= \begin{bmatrix} a+2c+d & -2a+b-c-8d \\ a-b-2c+5d & a+b+6c-2d \end{bmatrix}$$

3. Let S_{22} be the vector space of 2×2 symmetric matrices. Provide a basis for S_{22} so that the linear transformation R had a diagonal matrix representation. (20 points)

$$R\colon S_{22} \to S_{22}, \quad R\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = \begin{bmatrix} 2a - 5b + 5c & 5a - 8b + 5c \\ 5a - 8b + 5c & 5a - 5b + 2c \end{bmatrix}$$

$\dim(S_{22}) = 3$

$B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ matrix rep: $\begin{bmatrix} 2 & -5 & 5 \\ 5 & -8 & 5 \\ 5 & -5 & 2 \end{bmatrix} = M^R_{B,B}$

Eigenvalue | Eigenvector $\xrightarrow{\rho_B^{-1}}$ C

$\lambda = 2$ $\quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\lambda = -3$ $\quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$$M^R_{C,C} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

4. Consider the linear transformation from the first problem, T, reprinted below. (25 points)

$T\colon P_2 \to M_{12}, \quad T\left(a+bx+cx^2\right) = \begin{bmatrix}2a-3b+c & a-b+c\end{bmatrix}$

(a) Build a matrix representation of T relative to standard bases (denote these bases as X for the domain and Y for the codomain).

$X = \{1, x, x^2\}, \quad Y = \{[1\ 0], [0\ 1]\}$

With these choices we get matrix representation on sight,

$$M^T_{X,Y} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

(b) Build the matrices that relate the representation from part (a) of Problem 1 to the representation in part (a) of this problem, and do the computation that converts one representation into the other.

$$M^T_{B,C} = C_{Y,C}\, M^T_{X,Y}\, C_{B,X}$$

$$= C^{-1}_{C,Y}\, M^T_{X,Y}\, C_{B,X}$$

$$= \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 1 & 2 & 5 \\ 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -17 & -61 \\ 5 & 21 & 75 \end{bmatrix} \checkmark$$