Math 290 Name: Key Dr. Beezer
Exam 6 Fall 2013
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
As justification for your work, you may use Sage routines designed for matrices, but not routines designed for linear transformations.

1. Prove that the following function is a linear transformation. (15 points)

$T\colon \mathbb{C}^2 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+2b \\ a-\mathbf{b} \end{bmatrix}$ Two properties to check.

1) $T\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1+a_2 \\ b_1+b_2 \end{bmatrix}\right) = \begin{bmatrix} a_1+a_2+2(b_1+b_2) \\ a_1+a_2-(b_1+b_2) \end{bmatrix}$

$= \begin{bmatrix} a_1+2b_1 + a_2+2b_2 \\ a_1-b_1 + a_2-b_2 \end{bmatrix} = \begin{bmatrix} a_1+2b_1 \\ a_1-b_1 \end{bmatrix} + \begin{bmatrix} a_2+2b_2 \\ a_2-b_2 \end{bmatrix} = T\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right)$

2) $T\left(\alpha\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}\right) = \begin{bmatrix} \alpha a + 2(\alpha b) \\ \alpha a - \alpha b \end{bmatrix} = \begin{bmatrix} \alpha(a+2b) \\ \alpha(a-b) \end{bmatrix} = \alpha\begin{bmatrix} a+2b \\ a-b \end{bmatrix} = \alpha T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$

2. The linear transformation S is invertible (you may assume this). Find a formula for values of the inverse linear transformation, S^{-1}. (15 points)

$S\colon P_2 \to \mathbb{C}^3, \quad S\left(a+bx+cx^2\right) = \begin{bmatrix} -2a+b+c \\ -3a+b-c \\ -2a-3c \end{bmatrix}$

Compute preimages of a basis of the codomain.

$$\left.\begin{aligned} S^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) &= \{-3-7x+2x^2\} \\ S^{-1}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) &= \{3+8x-2x^2\} \\ S^{-1}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) &= \{-2-5x+x^2\} \end{aligned}\right\}$$

Via solving three linear systems. Each with the same nonsingular coefficient matrix.

Via theorem LTB,

$$S^{-1}(a+bx+cx^2) = aS^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + bS^{-1}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + cS^{-1}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= a(-3-7x+2x^2) + b(3+8x-2x^2) + c(-2-5x+x^2)$$

$$= (-3a+3b-2c) + (-7a+8b-5c)x + (2a-2b+c)x^2$$

3. In the following questions, analyze the linear transformation T. (40 points)

$$T\colon M_{22} \to P_2, \quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (-2a - 2b - 4c + 6d) + (b + c - 2d)x + (a + c - d)x^2$$

(a) Find a basis for the kernel of T, $\mathcal{K}(T)$.

$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 + 0x + 0x^2$

$\Rightarrow$ $-2a - 2b - 4c + 6d = 0$
$b + c - 2d = 0$
$a + c - d = 0$

coefficient matrix rref $\rightarrow$ $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So $a = -c + d$, $b = -c + 2d$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -c+d & -c+2d \\ c & d \end{bmatrix} = c\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

basis $= \left\{\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\right\}$ spans, linearly independent, see bottom rows

(b) Is T injective? Why or why not?

$K(T) \neq \{\underset{\sim}{0}\}$ so by Theorem KILT, T is not injective

(c) Find a basis for the range of T, $\mathcal{R}(T)$.

Theorem SSRLT, spanning set from outputs for basis of domain

$\left\{T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right\} = \{-2 + x^2, -2 + x, -4 + x + x^2, 6 - 2x - x^2\}$

This set is linearly dependent.

Usual techniques with systems of equations will allow us to write last two polynomials as linear combos of first two.

$-4 + x + x^2 = 1(-2 + x^2) + 1(-2 + x)$
$6 - 2x - x^2 = -1(-2 + x^2) + -2(-2 + x)$

basis $= \{-2 + x^2, -2 + x\}$

(d) Is T surjective? Why or why not?

$\dim(R(T)) = 2 \Rightarrow R(T) \neq P_2 \leftarrow$ dimension 3

So by Theorem RSLT, T is not surjective.

4. Suppose that $T\colon U \to V$ is a linear transformation. Prove Theorem LTTZZ: $T(\mathbf{0}) = \mathbf{0}$. Give a clear proof using words (not arrows) to indicate implications and giving a reason for every step. This will be graded partly on good style — a bare sequence of equations with no structure is unlikely to get full credit. (15 points)

$$
\begin{aligned}
\underset{\sim}{0} &= T(\underset{\sim}{0}) - T(\underset{\sim}{0}) && \text{Property AI (in } V) \\
&= T(\underset{\sim}{0}+\underset{\sim}{0}) - T(\underset{\sim}{0}) && \text{Property Z (in } U) \\
&= [T(\underset{\sim}{0})+T(\underset{\sim}{0})] - T(\underset{\sim}{0}) && \text{Definition LT} \\
&= T(\underset{\sim}{0}) + [T(\underset{\sim}{0}) - T(\underset{\sim}{0})] && \text{Property AA (in } V) \\
&= T(\underset{\sim}{0}) + \underset{\sim}{0} && \text{Property AI (in } V) \\
&= T(\underset{\sim}{0}) && \text{Property Z (in } V)
\end{aligned}
$$

5. Suppose that $T\colon U \to V$ and $S\colon V \to W$ are linear transformations. Prove the following relationship between ranges and composition: $\mathcal{R}(S \circ T) \subseteq \mathcal{R}(S)$. (15 points)

Suppose $\underset{\sim}{w} \in R(S \circ T)$. Then there exists $\underset{\sim}{u}$ (a preimage) so that $(S \circ T)(\underset{\sim}{u}) = \underset{\sim}{w}$. Rewritten, $S(T(\underset{\sim}{u})) = \underset{\sim}{w}$. Define $\underset{\sim}{v} = T(\underset{\sim}{u})$. Then $S(\underset{\sim}{v}) = \underset{\sim}{w}$. This is enough to say $\underset{\sim}{w} \in R(S)$.

So by Definition SSET, $R(S \circ T) \subseteq R(S)$.