1. Prove that the following function is a linear transformation. (15 points)

\[ T: \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} a + 2b \\ a - b \end{bmatrix} \]

2. The linear transformation \( S \) is invertible (you may assume this). Find a formula for values of the inverse linear transformation, \( S^{-1} \). (15 points)

\[ S: P_2 \rightarrow \mathbb{C}^3, \quad S \left( a + bx + cx^2 \right) = \begin{bmatrix} -2a + b + c \\ -3a + b - c \\ -2a - 3c \end{bmatrix} \]
3. In the following questions, analyze the linear transformation $T$. (40 points)

$T: M_{22} \rightarrow P_2, \quad T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (-2a - 2b - 4c + 6d) + (b + c - 2d)x + (a + c - d)x^2$

(a) Find a basis for the kernel of $T, \mathcal{K}(T)$.

(b) Is $T$ injective? Why or why not?

(c) Find a basis for the range of $T, \mathcal{R}(T)$.

(d) Is $T$ surjective? Why or why not?
4. Suppose that $T : U \to V$ is a linear transformation. Prove Theorem LTTZZ: $T(\mathbf{0}) = \mathbf{0}$. Give a clear proof using words (not arrows) to indicate implications and giving a reason for every step. This will be graded partly on good style — a bare sequence of equations with no structure is unlikely to get full credit. (15 points)

5. Suppose that $T : U \to V$ and $S : V \to W$ are linear transformations. Prove the following relationship between ranges and composition: $\mathcal{R}(S \circ T) \subseteq \mathcal{R}(S)$. (15 points)