Math 290 Name: Key Dr. Beezer
Exam 5 Fall 2013
Chapters D and E

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Instructions on the use of Sage are part of each individual problem.

1. Compute the determinant of the following matrix by expanding about the second column. Sage may not be used to justify your answer, nor will credit be given for expanding about some other row or column. (15 points)

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 2 \\ 4 & 1 & -2 \end{bmatrix}$$

$$\det(A) = (-1)(-1)\begin{vmatrix} 3 & 2 \\ 4 & -2 \end{vmatrix} + (+1)(2)\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} + (-1)(1)\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}$$

$$= (-14) + (-32) + (5) = -41$$

2. Determine all of the eigenspaces of the following matrix. Sage may not be used to justify any part of your answer, so you will need to do these computations "by-hand." (15 points)

$$B = \begin{bmatrix} 8 & -18 \\ 3 & -7 \end{bmatrix}$$

$$p_B(\lambda) = \begin{vmatrix} 8-\lambda & -18 \\ 3 & -7-\lambda \end{vmatrix} = (8-\lambda)(-7-\lambda) - 3(-18)$$

$$= -56 - \lambda + \lambda^2 + 54 = \lambda^2 - \lambda + 2 = (\lambda-2)(\lambda+1)$$

Eigenvalues: $\lambda = 2, \quad \lambda = -1$

$\underline{\lambda = 2}$ $\quad B - 2I_2 = \begin{bmatrix} 6 & -18 \\ 3 & -9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$

$$E_B(2) = N(B - 2I_2) = \left\langle\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$\underline{\lambda = -1}$ $\quad B - (-1)I_2 = \begin{bmatrix} 9 & -18 \\ 3 & -6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$$E_B(-1) = N(B + I_2) = \left\langle\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}\right\rangle$$

3. Perform the requested computations for the matrix C below. Each part has specific instructions on the use of Sage. (40 points)

$$C = \begin{bmatrix} 3 & 2 & 2 & 0 & 2 \\ -4 & 7 & -4 & 12 & -8 \\ 4 & -4 & 7 & -12 & 8 \\ 2 & 0 & 4 & -3 & 6 \\ 0 & 2 & 2 & 0 & 5 \end{bmatrix}$$

(a) Use Sage to find the characteristic polynomial (as a polynomial), then use Sage to factor the polynomial. From this, determine the eigenvalues of C and their algebraic multiplicities.

.charpoly() $\Rightarrow$

$$p_C(x) = x^5 - 19x^4 + 142x^3 - 522x^2 + 945x - 675$$

.factor() (or .fcp()) $\Rightarrow$

$$= (x-5)^2 (x-3)^3$$

So, $\alpha_C(5) = 2$, $\alpha_C(3) = 3$

(b) Choose one of the eigenvalues (your choice) and compute the eigenspace, using Sage only to form, and row-reduce, the relevant matrix.

$\lambda = 5$ could be the easier one, max 2-D eigenspace

$$(C-5).\text{rref}() \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad E_C(5) = \left\langle \left\{ \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

Theorem BNS

(c) Use the output of Sage's `.eigenspaces()` command to show that C is diagonalizable, being certain to indicate what theorem(s) you are employing.

We get dimensions of eigenspaces from this command

$$2 = \alpha_C(5) = \gamma_C(5) = 2$$
$$3 = \alpha_C(3) = \gamma_C(3) = 3$$

By Theorem DMFE, we now know C is diagonalizable.

(d) Find a nonsingular matrix S so that $S^{-1}CS$ is a diagonal matrix. Describe the columns of S. You may use any Sage commands you wish for this problem.

From .eigenmatrix_right();

$$S = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1/2 & 1/3 & -1 & -1/3 \\ 1 & 0 & 0 & -1 & -1 \end{bmatrix}$$

the columns of S are

1) A linearly independent set (Theorem DC)

2) Eigenvectors of C (Theorem DC)

4. Suppose that A is a nonsingular matrix and λ is an eigenvalue of A. Prove that $1/\lambda$ is an eigenvalue of A^{-1}. (15 points)

Let $\underset{\sim}{x}$ be an eigenvector of A.

Note A nonsingular $\Rightarrow \lambda \neq 0 \Rightarrow 1/\lambda$ is "ok".

$$A^{-1}\underset{\sim}{x} = \left(\frac{1}{\lambda}\lambda\right)A^{-1}\underset{\sim}{x}$$
$$= \frac{1}{\lambda}A^{-1}(\lambda \underset{\sim}{x})$$
$$= \frac{1}{\lambda}A^{-1}(A\underset{\sim}{x})$$
$$= \frac{1}{\lambda}(A^{-1}A)\underset{\sim}{x}$$
$$= \frac{1}{\lambda}I\underset{\sim}{x} = \frac{1}{\lambda}\underset{\sim}{x}$$

This says that $\underset{\sim}{x}$ is an eigenvector of A^{-1} for the eigenvalue $1/\lambda$.

5. Prove the following result about the effect of the second row operation on the determinant. Construct your proof from definitions and basic theorems about determinants, rather than using results about elementary matrices and row operations that are logically a consequence of this result.

Suppose that A is a square matrix and B is the matrix obtained from A by multiplying a single row by the scalar α. Then $\det(B) = \alpha\det(A)$. (15 points)

Suppose we multiply row k by α. So $[B]_{kj} = \alpha[A]_{kj}$ $\quad 1 \leq j \leq n$

Expand B about row k, Theorem DER,

$$\det(B) = \sum_{j=1}^{n} (-1)^{k+j}[B]_{kj}\det(B(k|j))$$
$$= \sum_{j=1}^{n} (-1)^{k+j}\alpha[A]_{kj}\det(B(k|j))$$

delete row k & submatrices are equal.

factor α out of scalar sum

$$= \alpha\sum_{j=1}^{n} (-1)^{k+j}[A]_{kj}\det(A(k|j))$$
$$= \alpha \underbrace{\det(A)}_{\text{by Theorem DER}}$$