Math 290 Name: Key Dr. Beezer
Exam 4 Fall 2013
Chapter VS

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use Sage to row-reduce matrices and to multiply matrices, so long as you explain your input and show your output in your work. No other use of Sage may be used as justification for your answers.

1. Determine if the following set, T, as a subset of the vector space $M_{2,3}$, is linearly independent or not. (~~10~~ 15 points)

$$T = \left\{ \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 2 & -4 & -1 \end{bmatrix}, \begin{bmatrix} -5 & 4 & -3 \\ 2 & -8 & 2 \end{bmatrix} \right\} = \{M_1, M_2, M_3\}$$

RLD $\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 = \mathcal{O}$

$$\begin{bmatrix} \alpha_1 - 5\alpha_3 & -\alpha_1 + \alpha_2 + 4\alpha_3 & \alpha_1 - \alpha_2 - 3\alpha_3 \\ -\alpha_1 + 2\alpha_2 + 2\alpha_3 & 3\alpha_1 - 4\alpha_2 - 8\alpha_3 & -\alpha_2 + 2\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Matrix equality gives homogeneous system

$$\begin{bmatrix} 1 & 0 & -5 \\ -1 & 1 & 4 \\ 1 & -1 & -3 \\ -1 & 2 & 2 \\ 3 & -4 & -8 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

coefficient matrix

The <u>only</u> RLD has $\alpha_1 = \alpha_2 = \alpha_3 = 0$
So T <u>is</u> linearly independent.

2. Prove that U is a subspace of P_2, the vector space of polynomials of degree 3 or less. (15 points)
$U = \{ a + bx + cx^2 \mid 3a + 6b - 3c = 0 \}$.

① $3(0) + 6(0) - 3(0) = 0$ so $0 + 0x + 0x^2 \in U$ and $U \neq \emptyset$.

② Suppose $a_1 + b_1x + c_1x^2,\ a_2 + b_2x + c_2x^2 \in U$.
Know $3a_1 + 6b_1 - 3c_1 = 0$ & $3a_2 + 6b_2 - 3c_2 = 0$.
$(a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2) = (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2$
Consider $3(a_1 + a_2) + 6(b_1 + b_2) - 3(c_1 + c_2)$
$= (3a_1 + 6b_1 - 3c_1) + (3a_2 + 6b_2 - 3c_2) = 0 + 0 = 0$
So sum <u>is</u> in U.

③ Suppose $\alpha \in \mathbb{C}$, $a_1 + b_1x + c_1x^2 \in U$
$\alpha(a_1 + b_1x + c_1x^2) = (\alpha a_1) + (\alpha b_1)x + (\alpha c_1)x^2$
Consider $3(\alpha a_1) + 6(\alpha b_1) - 3(\alpha c_1) = \alpha(3a_1 + 6b_1 - 3c_1) = \alpha \cdot 0 = 0$
So scalar multiple <u>is</u> in U. By Theorem TSS, U <u>is</u> a subspace of P_2.

3. U is identical to the subspace from the previous question. Show that the dimension of U is 2, by exhibiting a basis for U, being certain to establish carefully that the basis is both a spanning set for U and is linearly independent. (~~20~~ 15 points)

$U = \{\, a + bx + cx^2 |\, 3a + 6b - 3c = 0\}$ $\quad 3a+6b-3c \Rightarrow a = -2b+c$

$a+bx+cx^2 = (-2b+c)+bx+cx^2 = b(-2+x)+c(1+x^2)$

This shows that $S=\{-2+x,\ 1+x^2\}$ spans U.

RLD $\alpha_1(-2+x)+\alpha_2(1+x^2)=\mathbf{0}$

$(-2\alpha_1+\alpha_2)+\alpha_1 x+\alpha_2 x^2 = 0+0x+0x^2$

$\Rightarrow \alpha_1=\alpha_2=0.$

So S is linearly independent.

S is a basis of U with size 2, so $\dim(U)=2$.

4. U is identical to the subspace from the previous questions. Now assume that you know the dimension of U is 2. Answer the following with careful explanations of your answer. (~~30~~ points) 25

$U = \{\, a + bx + cx^2 |\, 3a + 6b - 3c = 0\}$

(a) Does the set $R = \{4 - x + 2x^2\}$ span U?

size of $R = 1 < 2 = \dim(U)$

So Theorem G says R does not span U.

(b) Is the set $S = \{-1 + 2x + 3x^2,\ 3 - x + x^2,\ -8 + 3x - 2x^2\} \subseteq U$ linearly independent?

Size of $S = 3 > 2 = \dim U$

So Theorem G says S is linearly dependent.

(c) Is the set $T = \{-5 + 3x + x^2,\ -8 + 5x + 2x^2\}$ a basis of U?

$\alpha_1(-5+3x+x^2)+\alpha_2(-8+5x+2x^2)=\mathbf{0}$

$(-5\alpha_1-8\alpha_2)+(3\alpha_1+5\alpha_2)x+(\alpha_1+2\alpha_2)x^2 = 0+0+x^2$

Homogenous system with coefficient matrix

$$\begin{bmatrix} -5 & -8 \\ 3 & 5 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \alpha_1=\alpha_2=0$$

So T is linearly independent.

size of $T = 2 = \dim(U)$ + T lin. ind. + Theorem G yields T spans U.

5. Suppose that $S = \{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ is a linearly independent subset of a vector space V. Prove that T is also a linearly independent subset of V. (15 points)

$T = \{\mathbf{u} - \mathbf{v} + \mathbf{w},\ \mathbf{u} + \mathbf{v} - 4\mathbf{w},\ 2\mathbf{u} + \mathbf{v} - 6\mathbf{w}\}$

RLD $\quad \alpha_1(\mathbf{u}-\mathbf{v}+\mathbf{w}) + \alpha_2(\mathbf{u}+\mathbf{v}-4\mathbf{w}) + \alpha_3(2\mathbf{u}+\mathbf{v}-6\mathbf{w}) = \mathbf{0}$

$(\alpha_1+\alpha_2+2\alpha_3)\mathbf{u} + (-\alpha_1+\alpha_2+\alpha_3)\mathbf{v} + (\alpha_1-4\alpha_2-6\alpha_3)\mathbf{w} = \mathbf{0}$

S linearly independent $\Rightarrow$

$\alpha_1+\alpha_2+2\alpha_3 = 0$

$-\alpha_1+\alpha_2+\alpha_3 = 0 \quad \Longrightarrow \quad \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\alpha_1 - 4\alpha_2 - 6\alpha_3 = 0$

(by usual techniques)

$\Rightarrow$ T is linearly independent

6. Suppose that $\alpha \in \mathbb{C}$, V is a vector space, $\mathbf{v} \in V$ and $\alpha\mathbf{v} = \mathbf{0}$. Prove that either $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$. (Notice that this is Theorem SMEZV so you are being asked to do more than just quote the theorem.) (15 points)

<u>Case 1</u> $\quad \alpha = 0$. Done.

<u>Case 2</u> $\quad \alpha \neq 0$. Then

$\mathbf{v} = 1\mathbf{v}$ Property O

$= \left(\frac{1}{\alpha}\alpha\right)\mathbf{v}$ Property MICN

$= \frac{1}{\alpha}(\alpha\mathbf{v})$ Property SMA

$= \frac{1}{\alpha}\mathbf{0}$ Hypothesis

$= \mathbf{0}$ Theorem ZVSM