Math 290 Name: Key Dr. Beezer
Exam 3 Fall 2013
Chapter M

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use Sage to row-reduce matrices and to multiply matrices, so long as you explain your input and show your output in your work. No other use of Sage may be used as justification for your answers.

1. Compute the inverse of the matrix A below, if it exists. You may use Sage, subject to the restrictions above. (15 points)

$$A = \begin{bmatrix} 5 & -7 & 7 & 4 \\ 7 & -8 & 9 & 5 \\ 3 & -4 & 4 & 3 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

Augment with the size 4 identity matrix and row-reduce with Sage.

$$[A|I_4] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -2 & 1 & 2 & -3 \\ 0 & 0 & 1 & 0 & -2 & 1 & 3 & -6 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 & -1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & -3 & 5 \\ -2 & 1 & 2 & -3 \\ -2 & 1 & 3 & -6 \\ -1 & 0 & 2 & -1 \end{bmatrix}$$

So by CINM + OSIS, or half of the proof of NI, the inverse is the final four columns

2. The coefficient matrix of the system below is the matrix A from the previous problem. *Use your answer to the previous question* to
(a) *first*, determine how many solutions the system has, and
(b) *second*, determine the solution set of the system.
Label parts (a) and (b) clearly in your answer, answer (a) without reference to (b), and be certain to base your answers on the results of the previous problem. There will be no partial credit for answers obtained by other approaches. (15 points)

$$\begin{aligned} 5x_1 - 7x_2 + 7x_3 + 4x_4 &= 16 \\ 7x_1 - 8x_2 + 9x_3 + 5x_4 &= 20 \\ 3x_1 - 4x_2 + 4x_3 + 3x_4 &= 13 \\ x_1 - x_2 + x_3 + x_4 &= 5 \end{aligned}$$

(a) By Theorem NI, the coefficient matrix is nonsingular since we found an inverse in the previous problem. By NMUS we know the solution is unique.

(b) Theorem SNCM tells us the solution is:

$$\underset{\sim}{x} = A^{-1}\underset{\sim}{b} = \begin{bmatrix} 1 & 0 & -3 & 5 \\ -2 & 1 & 2 & -3 \\ -2 & 1 & 3 & -6 \\ -1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 16 \\ 20 \\ 13 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 5 \end{bmatrix}$$

with Sage

3: Answer the following questions about the matrix B below. (40 points)

$$B = \begin{bmatrix} 7 & 12 & -18 & -61 & 27 \\ 4 & 2 & -5 & -15 & 11 \\ -1 & -3 & 4 & 14 & -5 \\ 0 & -1 & 1 & 4 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad D = \{1, 2, 3\}$$

(a) For the column space of B, find a linearly independent set S, with $\mathcal{C}(\underset{B}{\cancel{A}}) = \langle S \rangle$ and S contains only columns of B.

Theorem BCS says just take columns with indices in D.

$$S = \left\{ \begin{bmatrix} 7 \\ 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 12 \\ 2 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} -18 \\ -5 \\ 4 \\ 1 \end{bmatrix} \right\}$$

(b) For the row space of B, find a linearly independent set T, with $\mathcal{R}(\underset{B}{\cancel{A}}) = \langle T \rangle$.

Theorem BRS says to grab non zero rows of row equivalent matrix in reduced row-echelon form

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(c) For the column space of B, *use a technique substantially different from part (a)* to find a new linearly independent set R, with $\mathcal{C}(\underset{B}{\cancel{A}}) = \langle R \rangle$.

Analyzing the row space of the transpose is different.

$$B^t \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply BRS

$$R = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix} \right\}$$

$$\mathcal{C}(B) \overset{\text{CSRST}}{=} \mathcal{R}(B^t) = \langle R \rangle$$

(d) Find a non zero vector $\mathbf{c}$ so that the system $\mathcal{LS}(B, \mathbf{c})$ is consistent. Give an explanation of how you know the system is consistent *without simply solving the system.*

The "basis" in (c) is perfect for this. Any linear combination, even a simple one will satisfy CSCS.

$$\underset{\sim}{c} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Theorem CSCS is reason enough. Solving is a check.

(e) Find a vector $\mathbf{d}$ so that the system $\mathcal{LS}(B, \mathbf{d})$ is inconsistent. Give an explanation of how you know the system is inconsistent *without simply attempting to solve the system.*

Any linear combination of vectors in R will dictate the 4th entry. Perturbing it will put $\underset{\sim}{d}$ outside the column space so by CSCS system is inconsistent.

$$\underset{\sim}{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \text{ so}$$

we form $\underset{\sim}{d} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 672 \end{bmatrix}$

4. Suppose that $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$. Give a careful proof that $\alpha(A+B) = \alpha A + \alpha B$. (Notice that this is Propery DMAM so you are being asked to do more than just quote the statement of the property.) (15 points)

Use indexing to prove this matrix equality.

For $1 \le i \le m$, $1 \le j \le n$

$$
\begin{aligned}
[\alpha(A+B)]_{ij} &= \alpha [A+B]_{ij} && \text{Defn MSM} \\
&= \alpha([A]_{ij} + [B]_{ij}) && \text{Defn MA} \\
&= \alpha[A]_{ij} + \alpha[B]_{ij} && \text{Property DCN} \\
&= [\alpha A]_{ij} + [\alpha B]_{ij} && \text{Defn MSM} \\
&= [\alpha A + \alpha B]_{ij} && \text{Defn MA}
\end{aligned}
$$

By Definition ME, we see $\alpha(A+B) = \alpha A + \alpha B$

5. Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices. Prove that $A(B+C) = AB + AC$. (Notice that this is Theorem MMDAA so you are being asked to do more than just quote the statement of the theorem.) (15 points)

Indexing again, but with EMP. For $1 \le i \le m$, $1 \le j \le p$.

$$
\begin{aligned}
[A(B+C)]_{ij} &= \sum_{k=1}^{n} [A]_{ik} [B+C]_{kj} && \text{EMP} \\
&= \sum_{k=1}^{n} [A]_{ik} ([B]_{kj} + [C]_{kj}) && \text{MA} \\
&= \sum_{k=1}^{n} ([A]_{ik}[B]_{kj} + [A]_{ik}[C]_{kj}) && \text{Property DCN} \\
&= \sum_{k=1}^{n} [A]_{ik}[B]_{kj} + \sum_{k=1}^{n} [A]_{ik}[C]_{kj} && \text{Property CACN} \\
&= [AB]_{ij} + [AC]_{ij} && \text{EMP} \\
&= [AB+AC]_{ij} && \text{MA}
\end{aligned}
$$

So by Defn ME, $A(B+C) = AB + AC$.