Artinian and Noetherian Rings

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1 Introduction

Our study of ring theory in class covered a wide variety of areas in and uses of the topic. Field theory and polynomial rings were of particular interest with the end goal being Galois theory. One topic that was briefly introduced was Noetherian and Artinian rings. These two characterizations for rings are worth deeper study.

In a sense, Artinian and Noetherian rings have some measure of finiteness associated with them. In fact, the conditions for Artinian and Noetherian rings, called respectively the descending and ascending chain conditions, are often termed the minimum and maximum conditions. These properties make Artinian and Noetherian rings of interest to an algebraist. Furthermore, these two types of rings are related. One major goal of this paper is to arrive at conditions for Noetherian rings to be Artinian and vice versa.

Another, lesser goal of this paper is to consider non-commutative rings and how this structure change affects conditions for Artinian and Noetherian rings. In particular, right and left ideals are introduced. These ideal types pop up in non-commutative rings. In commutative rings the two definitions coincide.

In this paper, I will introduce Noetherian and Artinian rings in detail, but the focus will be on connecting the two ring types. The second section gives background definitions and information. The third section develops a brief history of Artinian and Noetherian ring theory through example. Sections four and five are the major focus of this paper. Section four introduces conditions for a Noetherian ring to be Artinian. Section five considers Artinian rings in detail and culminates in the Hopkins-Levitzki Theorem that connects Artinian rings to Noetherian rings. Section six is meant as a fun extension of Artinian and Noetherian ring theory. This section introduces modules and connects them to Artinian and Noetherian rings. The final section provides a conclusion to the paper.

2 Background and Definitions

Before we discuss Noetherian and Artinian rings, it is important to introduce the concepts behind them. In particular, we need definitions for the maximum and minimum conditions that characterize the two types of rings. We will state these definitions in terms of rings, but they can be generalized to apply to other algebraic objects besides rings. In addition, the definitions depend on ideals. In class, our discussion was primarily limited to commutative rings. Because of this, all ideals we discussed were two sided. In non-commutative rings, this does not have to be the case. In order to solidify what is meant, we provide the following definition.

Definition 1. *Let R be a ring and I be a subring of R. For an element $r \in R$, let $Ir \equiv \{ar | a \in I\}$. I is a* ***right ideal*** *of R if $Ir \subset I$ for all $r \in R$.*

Left ideals can be defined in an entirely analogous manner by replacing right with left and Ir with rI. In a commutative ring, it is clear that if the condition for a right ideal is met, then the condition for a left ideal must also be met. In non-commutative rings, right and left ideals do not have to coincide and in general do not. The definitions below use the term ideal loosely. That is, the term ideal can refer to right, left, or two-sided ideals. However, it is only referring to one type of ideal at a time. That being said, we introduce the following definition.

Definition 2. *Let R be a ring. Let $I_1, I_2, \ldots$ be an arbitrary chain of ideals in R such that $I_1 \subset I_2 \subset \cdots$. If there exists an $N \in \mathbb{N}$ such that $I_n = I_N$ for $n \geq N$, then R is said to satisfy the* ***Ascending Chain Condition (ACC)****.*

ACC can be understood as a maximum condition on ideal chains in a ring R. A ring satisfying ACC has chains of ideals that always top out. Such a ring is called Noetherian. This name comes from the mathematician Emmy Noether, who will be discussed more later [1]. If ACC is met on right ideals, the ring is right Noetherian. If it is met on left ideals the ring is left Noetherian. The term Noetherian is reserved for rings that satisfy ACC on both right and left ideals. In commutative rings, all three of these conditions coincide.

Like ACC, there is a similar minimum condition for ideal chains in a ring. This condition is given in the definition below.

Definition 3. *Let R be a ring. Let $I_1, I_2, \ldots$ be an arbitrary chain of ideals in R such that $I_1 \supset I_2 \supset \cdots$. If there exists an $N \in \mathbb{N}$ such that $I_n = I_N$ for $n \geq N$, then R is said to satisfy the* ***Descending Chain Condition (DCC)****.*

The minimum condition provided by DCC is equivalent to saying that all ideal chains in a ring R bottom out. Rings satisfying DCC are called Artinian after mathematician Emil Artin [1]. As with ACC, the terms right and left Artinian come into play for right and left ideals respectively. Artinian rings meet DCC on left and right ideals. Once again, all three conditions coincide for commutative rings.

Notice the similarity between the ACC and DCC definitions. Together, they provide some sense of boundedness or finiteness for rings. It is also the case that Artinian and Noetherian rings share many properties. The remaining sections of this paper establish the similarities between these two types of rings, leading to conditions for their similarity. In particular, we will be building to the Hopkins-Levitzki Theorem and the results that stem from it.

3 A Brief History Through Example

In 1921, Emmy Noether introduced the ACC for the first time in mathematics literature [1]. She was considering ideals in commutative rings. Let us consider a brief example.

3.1 A Simple Example

Up to this point, we have not seen an example of a Noetherian ring. We will start with a basic example for a commutative ring. As a note, we will use the notation for left ideals when considering ideals in a commutative ring. Since all ideals are both left and right in a commutative ring, this notation is chosen for convenience.

Consider the ring $\mathbb{Z}$ under standard multiplication and addition (i.e. the operations learned in grade school). Clearly, $\mathbb{Z}$ is commutative under multiplication since we know standard multiplication to be commutative. Thus, we have a commutative ring.

It remains to be seen that $\mathbb{Z}$ is indeed Noetherian. We begin by noting that all ideals in $\mathbb{Z}$ are principal. That is, $\mathbb{Z}$ is a principal ideal domain. Once again, a proof can be found in [4]. Thus, any ideal I of $\mathbb{Z}$ takes on the form $a\mathbb{Z}$ where a is an integer. Let's investigate the ideal $24\mathbb{Z}$. This ideal consists of all positive and negative multiples of 24. We know that 12 divides 24. Thus, $12\mathbb{Z}$ must contain all the elements of $24\mathbb{Z}$. Similarly, 6 divides 12 and 3 divides 6. Since 3 is prime, only 1 and 3 divide it. This gives us the chain $24\mathbb{Z} \subset 12\mathbb{Z} \subset 6\mathbb{Z} \subset 3\mathbb{Z} \subset \mathbb{Z}$. It is easy to see that this chain tops out at $\mathbb{Z}$ in a finite number of inclusions.

Using this intuition, we can consider what happens in general. Let I be an ideal in $\mathbb{Z}$. Since every ideal in $\mathbb{Z}$ is principal, $I = a\mathbb{Z}$ where a is some integer. In other words, I is the set of all positive and negative multiples of a. As we saw above, ideals generated by divisors of a will contain I. In fact, these are the only ideals that can contain I since all ideals have one generator. If a has divisors, we can choose one, call it b, and construct the ideal $b\mathbb{Z}$ where $a\mathbb{Z} \subset b\mathbb{Z}$.

The question now is, can we continue this process infinitely or will the chain eventually top out? To see that it indeed tops out we must consider the divisors of a. In order to have an infinite chain of proper ideal inclusions, we must have an infinite chain of divisors. Since $c\mathbb{Z} = -c\mathbb{Z}$ for any $c \in \mathbb{Z}$, we can consider just positive divisors. Doing so, it is easy to see that an infinite chain is impossible since for any integer c, its positive divisors must be less than or equal to $|c|$. Because c is just some integer, there are only a finite number of positive numbers less than $|c|$. Thus, any chain of proper ideal inclusions must top out. While this is by no means a rigorous proof, it gives us a good idea of how Noetherian rings and the ACC look in practice.

After Noether's introduction of the ACC, work in this area of ring theory exploded. Her results were expanded to non-commutative settings [1]. In addition, other similar conditions for ideals in a ring were introduced. In particular, Emil Artin formulated the DCC in 1927, which provided a minimum condition to complement the maximum condition given by the ACC. It was later discovered, first by Noether herself and then more formally by Hopkins and Levitzki, that the DCC is actually the stronger condition [1]. As a lead in to this, let's return to the ring of integers.

3.2 The Ring of Integers is not Artinian

The main purpose of this example is to show that Noetherian rings are not necessarily Artinian. Consider the ring of integers, $\mathbb{Z}$ from the previous example. We have already seen that ideal inclusion is intimately linked to division in this ring. For an ideal $I_1 = a\mathbb{Z}$ to contain an ideal $I_2 = b\mathbb{Z}$, a must divide b. For I_2 to contain a third ideal, $I_3 = c\mathbb{Z}$, b must divide c. If we continue this pattern, each successive ideal must have a generator that is a multiple of the generator of the ideal immediately previous in the chain. Since the integers are infinite, we can keep constructing multiples forever. Our chain will never bottom out. The integers fail to satisfy the DCC. Thus, $\mathbb{Z}$ is not Artinian.

4 Connecting Noetherian to Artinian

The integers provide us with an example that not all Noetherian rings are Artinian, but what else can we say about the relationship between Artinian and Noetherian rings? Are Artinian and Noetherian mutually exclusive? Are all Artinian rings Noetherian? Are there special conditions where Artinian and Noetherian intersect? Our next goal in this paper is to answer the above questions.

We begin with a lemma that connects prime ideals in commutative Noetherian rings and the DCC.

Lemma 4. *Let R be a commutative Noetherian ring. Then R satisfies the DCC on prime ideals [2].*

This fact is initially surprising. As we saw with $\mathbb{Z}$, a commutative Noetherian ring need not be Artinian. However, certain chains of ideals, namely prime ideals, satisfy the DCC. This is promising for our goal of connecting Artinian and Noetherian rings. Since all commutative Noetherian rings meet the DCC on prime ideals, it is natural to wonder if certain commutative Noetherian rings are also Artinian. In fact, lemma 4 inspires a theorem that gives necessary and sufficient conditions for a commutative Noetherian ring to be Artinian. This theorem is given below. For a proof of this theorem, see [2].

Theorem 5. *Let R be a commutative Noetherian ring. R is Artinian if and only if all prime ideals in R are maximal [2].*

Theorem 5 gives us deeper insight into our example with the integers. The ring of integers is a commutative Noetherian ring but is not Artinian. This means that not all prime ideals in $\mathbb{Z}$ are maximal. We will show this below.

Let I be a non-trivial ideal in $\mathbb{Z}$. Since the ring of integers is a principal ideal domain, $I = a\mathbb{Z}$ where a is some integer other than zero or one. Another way of stating this is that the ideal I is "generated" by a. By generated, we do not mean that I is cyclic with generator a. Rather, for every $b \in I$ there exists $r \in \mathbb{Z}$ such that $b = ar$. Clearly, $a \in I$. This is enough for us to conclude that $I = a\mathbb{Z}$ is a prime ideal. Since we know $\{0\}$ and $\mathbb{Z}$ to be trivially prime, this implies that all ideals of $\mathbb{Z}$ are prime.

Now we just need to show that at least one ideal of $\mathbb{Z}$ is not maximal. We claim that any non-trivial ideal $c\mathbb{Z}$, where c is not prime, is not maximal. From our original example with the ring of integers, we know that there exists $d \in \mathbb{Z}$, $d \neq 1, d \neq c$, such that d divides c and $c\mathbb{Z} \subset d\mathbb{Z}$. Thus, $c\mathbb{Z}$ is not maximal in $\mathbb{Z}$. We conclude using Theorem 5 that $\mathbb{Z}$ is not Artinian.

We now have a characterization for Noetherian rings that are also Artinian. This covers half of the major question in this paper. Particular Noetherian rings are indeed Artinian. Now we want to examine the Noetherian and Artinian relationship from the perspective of Artinian rings. This position is developed in the next section.

5 The Artinian Perspective

In this section, we will simultaneously tackle two issues that have yet to be addressed. One issue revolves around the conditions for when an Artinian ring is Noetherian. As it turns out, all Artinian rings with identity are Noetherian. The second concern regards the fact that up to this point we have only considered commutative rings. However, non-commutative Artinian and Noetherian rings can also be linked. Both of these topics will be addressed in the Hopkins-Levitzki Theorem, introduced later. First, we introduce an example of an Artinian ring.

5.1 An Artinian Example

We have seen previously that $\mathbb{Z}$ is not an Artinian ring. However, it can be shown that a close relative of $\mathbb{Z}$ is Artinian. This relative happens to be the ring of rational numbers, which we will denote as $\mathbb{Q}$. We know $\mathbb{Q}$ to be a field since it is the field of fractions for $\mathbb{Z}$. For a more complete discussion of this see [4].

We want to show that $\mathbb{Q}$ is Artinian, so we need to consider the ideals of $\mathbb{Q}$. Since $\mathbb{Q}$ is a field, and thus commutative, its ideals are two-sided. Two ideals of $\mathbb{Q}$ are the trivial one, $\{0\}$, and $\mathbb{Q}$ itself. We will argue that these ideals are the only two.

Let I be a non-trivial ideal in $\mathbb{Q}$, and let $a \in I$. Since I is an ideal, it has the absorbing property. That is, $rI \subseteq I$ for all $r \in \mathbb{Q}$. This means that $ra \in I$ for any $r \in \mathbb{Q}$. Since $a \in I$, $a \in \mathbb{Q}$. This means a has an inverse, a^{-1}, because $\mathbb{Q}$ is a field. By the above, we get that $a^{-1}a = 1 \in I$. With the identity element of $\mathbb{Q}$ in I, it is easy to see that the absorbing property implies $I = \mathbb{Q}$. Thus, we have shown that any non-trivial ideal in $\mathbb{Q}$ must be $\mathbb{Q}$.

We can now state with confidence that $\mathbb{Q}$ has only two ideals, $\{0\}$ and $\mathbb{Q}$. Clearly, $\mathbb{Q} \supset \{0\}$. This finite chain of ideals clearly bottoms out. This is the only possible chain of proper ideal inclusions in $\mathbb{Q}$ since $\mathbb{Q}$ only has two ideals. Thus, $\mathbb{Q}$ is Artinian.

The field of rational numbers provides us with an example of an Artinian ring. It should be noted that by the same logic as in the preceding paragraph, $\mathbb{Q}$ is also Noetherian. In addition, we can easily see that $\mathbb{Q}$ satisfies Theorem 5. In $\mathbb{Q}$, the ideals $\{0\}$ and $\mathbb{Q}$ are trivially prime since $\{0\}$ contains just one element and $\mathbb{Q}$ contains all elements of the field of rational numbers. These ideals are also maximal because it is impossible to slip an ideal in between $\{0\}$ and $\{0\}$ or $\mathbb{Q}$ and $\mathbb{Q}$.

5.2 The Hopkins-Levitzki Theorem

The field of rational numbers provided us with an example of a commutative ring that is both Artinian and Noetherian. We have already considered this ring from the perspective that it is a Noetherian ring that satisfies conditions for being Artinian. What can we say if we view the field of rational numbers as an Artinian ring first? In general, we can conclude that any Artinian ring with identity is Noetherian.

Before formally stating this result, we will first give some background. It has been previously stated that Emmy Noether first introduced the ACC for commutative rings. Emil Artin later introduced the DCC and expanded his results to non-commutative rings. In 1929, Noether noticed that Artin did not need ACC in one of his proofs because it was implied by the DCC assumption [1]. Finally, in 1939, Hopkins and Levitzki independently discovered that the DCC is actually the stronger condition. For both mathematicians, the setting was ideals in non-commutative rings. Levitzki proved that all right Artinian rings with identity are right Noetherian [5]. Hopkins showed the same result for left Artinian rings and left Noetherian [3]. Together these results give that all Artinian rings with identity are Noetherian. This is summarized in the following theorem, named for both mathematicians.

Theorem 6. ***(Hopkins-Levitzki)*** *Let R be a right (left) Artinian ring with identity. Then R is right (left) Noetherian.*

The proof of this theorem requires a number of definitions not introduced in this paper. For those that are curious, an equivalent statement of the theorem and its proof can be found in [2] or Hopkins' and Levitzki's original papers [3,5].

6 Artinian, Noetherian, and Modules

Up through this point in the paper, we have progressed through history from Noether's work in 1921 to Hopkins-Levitzki in 1939. In a broad sense, we can now characterize Artinian and Noetherian rings in terms of one another. Artinian rings with identity are Noetherian. Also, commutative Noetherian rings are Artinian if and only if their prime ideals are maximal. While these results are important, we are left asking what has happened in this particular area of ring theory in the last seventy years.

The answer to this question lies in The Hopkins-Levitzki Theorem. The theorem originated as a result for rings. In the past seventy years, this theorem has been expanded in a number of ways, including latticial, categorical, and Krull dimension-like forms [1]. One interesting application of the the Hopkins-Levitzki Theorem is in the realm of modules. It is this application that we will discuss in this section.

The Artinian and Noetherian framework that we discussed above for rings can be generalized to encompass modules, another type of algebraic structure. In simple terms, modules are like vector spaces but over general rings instead of fields. This section of the paper is not intended to be an extensive introduction to modules. Rather, I will briefly define modules and submodules and then discuss their application to Artinian and Noetherian conditions.

We begin with the following definition for module.

Definition 7. *Let R be a ring with identity 1_R. M is a* ***left R-module*** *if there exists an addition operation on which M is an abelian group and a scalar multiplication operation $R \times M \to M$ such that for all $a, b \in R$ and $x, y \in M$ the following hold:*

- $a(x + y) = ax + ay$
- $(a + b)x = ax + bx$
- $(ab)x = a(bx)$
- $1_R x = x$.

A right module can be defined in an analogous manner with the ring acting on the right rather than the left.

Next we define submodule. The definition is very similar to the definition for ideals in rings.

Definition 8. *Let M be an R-module on a ring R. Suppose N is a subgroup of M. Then N is an* ***R-submodule*** *if N has the absorbing property on the left. That is, for every $n \in N$ and $r \in R$, $rn \in N$.*

With these definitions in hand, we can start generalizing our previous results for rings. To begin with, we can extend the ACC and DCC to modules by replacing ring with module and ideal with submodule. A module that satisfies the ACC is called a Noetherian module. A module that satisfies the DCC is called an Artinian module.

We can use this new setting to rederive our results for rings. One advantage of modules in this area is that we can construct rings by considering a module, that happens to be a ring, over itself. Recall that all rings are abelian groups with respect to addition, so we can make this statement. In this way, Artinian and Noetherian modules can be seen as a generalization of the ring results.

This section was meant to be a small taste of the importance of Artinian and Noetherian conditions in a broader algebraic sense. By generalizing Artinian and Noetherian rings to modules, mathematicians in recent history have been able to gain a deeper understanding of ring theory and finiteness conditions in algebra.

7 Conclusion

Artinian and Noetherian rings are far more similar than they appear on the surface. While Artinian rings satisfy a minimum condition and Noetherian rings satisfy a maximum condition, these conditions do coincide in several cases. Most prominently, all Artinian rings with identity are Noetherian, as given by the Hopkins-Levitzki Theorem. Equivalently, the DCC encompasses the ACC for rings with unity. The DCC turns out to be the stronger condition.

We have also seen that Noetherian and Artinian conditions can be generalized to modules. Although not extensively discussed in this paper, these module specific definitions give rise to a wide variety of expansions for the Hopkins-Levitzki Theorem, as well as Artinian and Noetherian theory in general. I hope that this paper has been an interesting introduction into Noetherian and Artinian rings and that it has given a decent idea of the power of the Artinian and Noetherian framework in algebra.

8 Bibliography

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