# Classifying All Groups of Order 16

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# 1 Introduction

In group theory, we are often interested in classifying all groups of a certain order by isomorphism class, which demonstrates that they have the same structure. For orders up to 15, we have already determined the isomorphism classes. This paper will extend our classification to the groups of order 16. To begin, we introduce some basic notation:

**Notation** *A group of order 16 will be denoted $G$. The symbol $\cong$ will stand for 'isomorphic'.*

In order to describe any group, the representation $G = \{a_1^{\alpha_1}a_2^{\alpha_2}...a_n^{\alpha_n} : a_1^{\beta_1} = a_2^{\beta_2} = ...a_n^{\beta_n} = e, a_2a_1 = a_1a_2a_{1,2}, a_3a_1 = a_1a_3a_{1,3}, ...a_na_{n-1} = a_{n-1}a_na_{n-1,n}\}$ will be used. This tells us the elements of $G$ in terms of generators and the orders of the generators. The representation also describes how the generators commute, so we can condense a string of the elements into the form used in the presentation. Two notes: I have left out a piece where, occasionally, powers of the generators equal each other when they are less than their orders (which would also be stated). Also, you cannot have just any old representation, but that is a different paper topic. We begin by noting that the prime factorization of 16 is $16 = 2^4$, so any group of order 16 is a $p$-group (a group whose order is the power of a prime, in this case 2). As such we will classify with factor groups. In particular, we will use the center of a group in our classification:

**Definition** *The center of any group is the set of all elements that commute with every element in a group, denoted $Z(G) = \{z : zg = gz, \forall g \in G\}$.*

We have the following two theorems about the center:

**Theorem 1.1** *For a p-group, The center of a group is a nontrivial subgroup (Judson, 186).*

**Theorem 1.2** *The center of a group is also a normal subgroup (Judson, 186).*

To classify the groups of order 16, we consider the different cases for the order of the center. Since $Z(G)$ is a nontrivial subgroup, $Z(G)$ must divide the order of the group, so the possibilities for $|Z(G)|$ are 16, 8, 4, and 2. Based on the center, we then build the factor group $G/Z(G)$, which will have order $|G/Z(G)| = |G|/|Z(G)|$. Then using the Correspondence Theorem, we deduce the properties of the group for various cases.

**Theorem 1.3 (Correspondence Theorem)** *If $N$ is a normal subgroup of any group $G$. Then $S \mapsto S/N$ is a one-to-one correspondence between the set of subgroups $S$ containing $N$ and the subgroups of $G/N$. Furthermore, the normal subgroups in $S$ correspond to normal subgroups in $G/N$, and if a subgroup of $G/N$ is contained in a subgroup of $G/N$, then the corresponding subgroups in $S$ share the same relation (Judson, 147).*

From this, we see that we need to know the groups of order 8, 4, and 2, shown in the table below.

| Name | Order | Symbol | Representation | Number and Structure of Nontrivial Subgroups | Center |
|---|---|---|---|---|---|
| Integers mod 2 | 2 | $\mathbb{Z}_2$ | $\{a^\alpha : a^2 = e\}$ | None | abelian |

| Name | Order | Symbol | Representation | Number and Structure of Nontrivial Subgroups | Center |
|---|---|---|---|---|---|
| Integers mod 4 | 4 | $\mathbb{Z}_4$ | $\{a^\alpha : a^4 = e\}$ | 1 isomorphic to $\mathbb{Z}_2$ | abelian |
| Klein 4 group | 4 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta : a^2 = b^2 = e, ba = ab\}$ | 3 isomorphic to $\mathbb{Z}_2$ | abelian |
| Integers mod 8 | 8 | $\mathbb{Z}_8$ | $\{a^\alpha : a^8 = e\}$ | 1 isomorphic to $\mathbb{Z}_4$ 1 isomorphic to $\mathbb{Z}_2$ | abelian |
| Direct Product of $\mathbb{Z}_4$ and $\mathbb{Z}_2$ | 8 | $\mathbb{Z}_4 \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta : a^4 = b^2 = e, ba = ab\}$ | 2 isomorphic to $\mathbb{Z}_4$ 1 isomorphic to $\mathbb{Z}_2$ $\times\mathbb{Z}_2$, 3 isomorphic to $\mathbb{Z}_2$ | abelian |
| Direct Product of $\mathbb{Z}_2$, $\mathbb{Z}_2$ and $\mathbb{Z}_2$ | 8 | $\mathbb{Z}_2 \times \mathbb{Z}_2$ $\times\mathbb{Z}_2$ | $\{a^\alpha b^\beta c^\gamma : a^2 = b^2 = c^2 = e,$ $ba = ab$, $ca = ac$, $cb = bc\}$ | 7 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, 7 isomorphic to $\mathbb{Z}_2$ | abelian |
| Quaternion Group | 8 | $\mathcal{Q}$ | $\{a^\alpha b^\beta : a^2 = b^2, a^4 = b^4 = e,$ $ba = a^{-1}b\}$ | 3 isomorphic to $\mathbb{Z}_4$ 1 isomorphic to $\mathbb{Z}_2$ | $\{e, a^2\} \cong$ $\mathbb{Z}_2$ |
| Dihedral Group | 8 | $D_4$ | $\{a^\alpha b^\beta : a^4 = b^2 = e,$ $ba = a^{-1}b\}$ | 1 isomorphic to $\mathbb{Z}_4$ 2 isomorphic to $\mathbb{Z}_2$ $\times\mathbb{Z}_2$, 5 isomorphic to $\mathbb{Z}_2$ | $\{e, a^2\} \cong$ $\mathbb{Z}_2$ |

Table 1: Groups of Order 2, 4, and 8 (Source: sagenb.org)

## 2 $|Z(G)| = 16$

To begin, assume $|Z(G)| = 16 = |G|$. Since a subset of a finite set equals the set if they have the same number of elements, $Z(G) = G$. Next, we note a fairly obvious fact:

**Theorem 2.1** *A group is abelian if and only if $Z(G) = G$*

Proof: If a group is abelian, then every element commutes with every element, so every element is in the center. Likewise, if the center equals the group, then every element is in the center and commutes with every element, and thus the group is abelian.

Since our group is abelian, we can use the Fundamental Theorem of Abelian Groups:

**Theorem 2.2 (Fundamental Theorem of Finite Abelian Groups)** *Every finite abelian group is isomorphic to a direct product of cyclic groups of the form $\mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times ...\mathbb{Z}_{p_n^{\alpha_n}}$, where the $p_i$ are (not necessarily distinct) primes (Judson, 172).*

Since the group is isomorphic to the direct product of cyclic groups, we note that the only possibilities for the order of cyclic groups are powers of 2. The sum of the powers must equal 4, so we have 5 ways of writing 4 as the sum of positive integers: 4=4, 4=3+1, 4=2+2, 4=2+1+1, and 4=1+1+1+1. Thus, there are five abelian isomorphism classes for the groups of order sixteen,

$$G \cong \mathbb{Z}_{2^4} = \mathbb{Z}_{16}, \quad G \cong \mathbb{Z}_{2^3} \times \mathbb{Z}_2 = \mathbb{Z}_8 \times \mathbb{Z}_2, \quad G \cong \mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} = \mathbb{Z}_4 \times \mathbb{Z}_4,$$
$$G \cong \mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

We will put these in the table below, with their representations.

| Name | Symbol | Representation | Center |
|---|---|---|---|
| Integers mod 16 | $\mathbb{Z}_{16}$ | $\{a^\alpha : a^{16} = e\}$ | abelian |
| Direct Product of $\mathbb{Z}_8$ and $\mathbb{Z}_2$ | $\mathbb{Z}_8 \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta : a^8 = b^2 = e, ba = ab\}$ | abelian |
| Direct Product of $\mathbb{Z}_4$ and $\mathbb{Z}_4$ | $\mathbb{Z}_4 \times \mathbb{Z}_4$ | $\{a^\alpha b^\beta : a^4 = b^4 = e, ba = ab\}$ | abelian |
| Direct Product of $\mathbb{Z}_4$, $\mathbb{Z}_2$ and $\mathbb{Z}_2$ | $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta c^\gamma : a^4 = b^2 = c^2 = e, ba = ab, ca = ac, cb = bc\}$ | abelian |
| Direct product of $\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_2$ and $\mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta c^\gamma d^\delta : a^2 = b^2 = c^2 = d^2 = e, ba = ab, ca = ac, da = ad, cb = bc, db = bd, dc = cd\}$ | abelian |

Table 2: Abelian Groups of Order 16

# 3 Using the Correspondence Theorem

From this point on, I will follow a basic method for determining the non-abelian groups. I will start by stating the order of the center, and then from this state the possibilities for the group based on the structure of the center and factor group. In doing so, I hope to either determine if a contradiction occurs, or find a representation for the group based on the elements needed to generate the group and how they commute. I will thus use the following notation and theorems, which apply regardless of the order of the center (any conditions for these theorems will be stated). To begin, we note that, as the center is a normal subgroup, the factor group will have subgroups. By the Correspondence Theorem, these subgroups contain the center, and some of these will be of order 8. We denote these subgroups, and prove a few theorems about them, below:

**Notation** *Denote the subgroups of order 8 in $G$ containing $Z(G)$ by $G_i$ where $i$ indexes the subgroup. Also, unless specified $z$ will denote an element of $Z(G)$, and $g_i$ will denote a element of $G_i$ not in $Z(G)$.*

**Theorem 3.1** *The center of $G_i$ contains the center of $G$, $Z(G) \subset \mathrm{Z}(G_i)$.*
Proof: This is trivial as an element is in the center commutes with every element, and thus commutes with all of $G_i$.

Next we state a useful theorem about the product set of two groups, one which we will use repeatedly:

**Theorem 3.2** *Let* H, K *be subgroups of $G$. Then the subset $HK = \{hk : h \in H, k \in K\}$ has order $|HK| = |H||K|/|H \cap K|$ (Judson, 203).*

From this, we see some useful facts about the $G_i$:

**Theorem 3.3** *The intersection of two $G_i$ must be a group of order 4.*
Proof: Let $G_i$, $G_j$ be distinct subgroups of order 8 containing the center. Then, by theorem 4.2, $|G_iG_j| = |G_i||G_j|/|G_i \cap G_j| = 8 * 8/|G_i \cap G_j| = 64/|G_i \cap G_j|$. Since the intersection of a subgroup is a subgroup (Judson, 46), the order of $G_i \cap G_j$ must be either 8,4,2, or 1. if $|G_i \cap G_j| = 8$, then the groups are the same, and for order 1 or 2, we get that $|G_iG_j|$ is 64 and 32, respectively. This leads to a contradiction, as then $G_iG_j$ is larger than $G$. Thus, the order of the intersection of two distinct $G_i$ is a group of order 4.

**Theorem 3.4** *Three distinct $G_i$ that share a common intersection (that will be a group of order 4 by Theorem 4.3) are formed by the cosets of their intersection. In particular $G_i = (G_i \cap G_j) \cup g_i(G_i \cap G_j)$, where $g_i$ is not in the intersection. If three distinct $G_i$ share a common intersection, then they contain every element in the group*

Proof: To show that this is true, note that both $(G_i \cap G_j), g_i(G_i \cap G_j)$ are subsets of $G_i$, and since $g_i$ is not in the intersection, they are disjoint. Hence there are eight elements in the union, which equals the number of elements in $G_i$, so the sets are equal. From this, having three $G_i$ means we have four distinct cosets (one that is the intersection and three for each $G_i$) so these are the number of cosets, and every element is in a coset by Lagrange's Theorem (Judson, 81), so every element is in at least one $G_i$ (as every element is in a coset of the intersection as a subgroup).

**Theorem 3.5** *Let $g_i$, $g_j$ be elements of $G_i$, $G_j$ with neither of them in $G_i \cap G_j$. Then $g_i g_j$ is not in $G_i$ or $G_j$.*

Proof: Without loss of generality, we will show $g_i g_j \notin G_i$. For the purposes of contradiction, assume that for $h_i \in G_i$, $h_i = g_i g_j$. Then $g_j = g_i^{-1} h_i$ is in $G_i$, which would mean $g_j$ is in the intersection of $G_i$ and $G_j$, which contradicts our assumptions about $g_j$. So $g_i g_j$ is not in either $G_i$ or $G_j$.

**Theorem 3.6** *Let $G/Z(G)$ be abelian. Then the commutator between $g_i$, $g_j$ is an element $z'$ of the center, $g_i g_j = z' g_j g_i$.*

Proof: let $g_i Z(G), g_j Z(G) \in G/Z(G)$. Then since the factor group is abelian, $(g_i Z(G))(g_j Z(G)) = (g_j Z(G))(g_i Z(G))$. Note that when we preform the operation in the factor group, we get $g_i g_j Z(G) = g_j g_i Z(G)$, so $g_i g_j = z' g_j g_i$ for some $z' \in Z(G)$ from the properties of cosets.

# 4 $|Z(G)| = 8$

In the case that $|Z(G)| = 8$, then we form the factor group $G/Z(G)$, which has order $|G/Z(G)| = |G|/|Z(G)| = 16/8 = 2$. Since the factor group is a group of order 2, the factor group is cyclic. However, we have a theorem about the factor group being cyclic:

**Theorem 4.1** *If $G/Z(G)$ is cyclic, then $G$ is abelian (Judson, 186).*

By theorem 2.1, a group that is abelian has a center equal to the group. If the center equals the group, then the order of the center is 16 and not 8. Hence we have a contradiction, so no groups of order 16 have a center of order 8.

# 5 $|Z(G)| = 4$

If the order of the center is four, then the order of $G/Z(G)$ is $|G/Z(G)| = |G|/|Z(G)| = 16/4 = 4$. The only possibilities for the factor group are thus $G/Z(G) \cong \mathbb{Z}_4$ or $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. If $G/Z(G) \cong \mathbb{Z}_4$, then the factor group is cyclic, which as we saw above implies a center of order sixteen, not four. Hence the factor group must be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. We note that, in this case, there are three subgroups of order 2, and by the Correspondence Theorem, there are three $G_i$ (the mapping from $G$ to $G/Z(G)$ is a 4 to 1 map, hence a subgroup of order 2 in the factor group is of order $2 * 4 = 8$ in $G$). We also note that by Theorem 3.3, the intersection of two $G_i$ is the center, as it is in the intersection and a subgroup of order 4.

We next note that, with $|Z(G)| = 4$ the only possibilities are that all $G_i$ are abelian, since the by theorem 3.1, the center of $G_i$, would have at least four elements in $Z(G_i)$, and not two as with the non-abelian groups of order 8. Hence $G_i$ are $\cong \mathbb{Z}_8$ , $\mathbb{Z}_4 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We now note a fact about commuting two elements

**Theorem 5.1** *If $|Z(G)| = 4$, and $g_i$, $g_j$ with $i \neq j$, then they do not commute.*

Proof: If $g_i, g_j$ commute when they are in different $G_i$, then $g_i$ commutes with all elements in $Z(G)$ (as the center), $g_i Z(G)$ (as commutes with itself and the center) $g_j Z(G)$ (as this is $g_j$ times an element of the center). This means there are at least 12 elements in the centralizer of $g_i$ (the centralizer of $g_i$ is the collection of all elements that commute with $g_i$), and since the centralizer is a subgroup (Judson, 185), the order of the centralizer must divide the order of the group. So the centralizer is of order 16 and hence the group. Then $g_i$ commutes with all elements, so it is in the center, which contradictions the fact that it is in the center. hence $g_i, g_j$ do not commute, so there commutator is a nontrivial element of the center.

We are now ready to classify groups. Since the center is of order 4, it is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. We will break these into two cases.

## 5.1 $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Since $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we know that each $G_i$ contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since all $G_i$ are abelian , and we know that $\mathbb{Z}_8$ contains no subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the only possible isomorphism classes for the three $G_i$ are $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. We now have four possibilities; (5.1.1) all $G_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, (5.1.2) two $G_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, (5.1.3) one $G_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and (5.1.4) no $G_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (so all $\cong \mathbb{Z}_4 \times \mathbb{Z}_2$). Considering the situations:

### 5.1.1 $G_1, G_2, G_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

To investigate the scenario of all $G_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we need the following theorem:

**Theorem 5.1.1.2** *If every nonidentity element in G has order 2, then G is abelian (Judson, 46).*

From theorem 3.4, every element is in a $G_i$. If all $G_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ then every non-identity element has order 2 (as every element is in a subgroup with nonidentity elements having order 2), and from Judson, we see that this implies that the group is abelian, so $|Z(G)| = 16 \neq 4$, so this leads to a contradiction. Hence, there are no groups with this property.

### 5.1.2 $G_1, G_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $G_3 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$

We will now state conditions that we will use over and over again, so we will call them:

**Conditions** [1] let $g_i, \in G_i$, $g_j \in G_j$ be elements with the property that $g_i, g_j \notin G_i \cap G_j$. Let $g'$ be the commutator between $g_j$, $g_i$ (this means $g_j g_i = g' g_i g_j$). We note that, for $g' \in Z(G)$ $(g_i g_j)^2 = g_i(g_j g_i)g_j = g_i g' g_i g_j^2 = g' g_i^2 g_j^2$.

Pick an element $r \in G_3$, $s \in G_1$ with conditions 1 such that $|r| = 4$ and $|s| = 2$ (we have these elements since there is a cyclic subgroup of order 4 in $G_3$ and a subgroup $G_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ that is not the center). Let $z'$ be the commutator between $s$ and $r$ (we will see the reason for calling these elements $r, s$ shortly). We note that $r^2 \in Z(G)$, as $|r^2| = 2$ and all elements of order 2 in $G_3$ are in the center (property of $\mathbb{Z}_4 \times \mathbb{Z}_2$). We note that $rs \in G_2$, so its order is 2 (theorem 4.4), and thus $e = (rs)^2 = zr^2s^2 = zr^2$. Hence $z = (r^2)^{-1} = r^2$, so $sr = r^2rs = r^3s = r^{-1}s$. Thus we can form the subgroup $H = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\} \subset G$. We see that $H$ is a subgroup as $H$ is closed (using the commuting rule) and has inverses (elements in $\langle r\rangle$ have inverses in $\langle r\rangle$ and elements not in $\langle r\rangle$ have order 2). If we choose $z \in Z(G)$ so $z \neq e, r^2$, then $\{z, e\} = \langle z\rangle = K$ is a subgroup. We note that $H \cap K = \{e\}$, so $|HK| = |H||K|/|H \cap K|$=8*2/1=16, so $HK = G$, and since $K$ is a subgroup of the center $hk = kh$ for all $k \in K, h \in H$. Hence, $G$ is the direct internal product of $H \cong D_4$ and $K \cong \mathbb{Z}_2$, so $G \cong H \times K \cong D_4 \times \mathbb{Z}_2$. This group has representation $G = \{r^\alpha s^\beta z^\gamma : r^4 = s^2 = z^2 = e, sr = r^{-1}s, zr = rz, zs = sz\}$.

### 5.1.3 $G_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $G_2, G_3 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$

In this case pick elements $g_2$, $g_3$, with conditions [1], which since they are not in the center must have order 4. Let $z'$ be the commutator between $g_3$ and $g_2$. we know that the elements squared are in the center, so let $g_2^2 = z_2$, $g_3^2 = z_3$. We note that $g_2g_3 \in G_1$, so $|g_2g_3| = 2$. This means that $e = (g_2g_3)^2 = z'g_2^2g_3^2 = z'z_2z_3$. We next prove a fact about $\mathbb{Z}_2 \times \mathbb{Z}_2$:

**Theorem 4.1.3.1** *let $a, b, c$ be nonidentity elements in $\mathbb{Z}_2 \times \mathbb{Z}_2$. $abc = e$, if and only if none of the elements are equal*

Proof: ($\Leftarrow$) If $c \neq a$, $c \neq b$, and $c \neq e$, then the only element left is $c = ab$. Consequently, $abc = cc = c^2 = e$.

($\Rightarrow$) We will prove the contrapositive so let two of the elements be equal and we want to show $abc \neq e$. Without loss of generality assume $a = b$. then $abc = a^2c = c \neq e$, so since the contrapositive is true, the statement is true, so $abc = e$ if and only if $a \neq b \neq c$

From the above theorem, we see that $z' \neq z_2 \neq z_3$, so $z' = z_2z_3$ from the properties of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, we get that $g_3g_2 = (z_2z_3)g_2g_3 = z_2g_2z_3g_3 = g_2^3g_3^3 = g_2^{-1}g_3^{-1}$. Next, we note that $\langle g_2\rangle = \{e, g_2, z_2, z_2g_2\}$ and $\langle g_3\rangle = \{e, g_3, z_3, z_3g_3\}$, so $\langle g_2\rangle \cap \langle g_3\rangle = \{e\}$. Thus $|\langle g_2\rangle\langle g_3\rangle| = |\langle g_2\rangle||\langle g_3\rangle|/|\langle g_2\rangle \cap \langle g_3\rangle| = 4 * 4/1 = 16$. Thus we have 16 distinct products of the form $g_2^\alpha g_3^\beta$, so group has representation $G = \{g_2^\alpha g_3^\beta : g_2^4 = g_3^4 = e, g_3g_2 = g_2^{-1}g_3^{-1}\}$ (this is called semidirect product of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$, $G = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$).

### 5.1.4 $G_1, G_2, G_3 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$

Grab $i \in G_1$, $j \in G_2$ with conditions [1], so since they not in the center, so $|i| = |j| = 4$. Their squares are in the center (property of $\mathbb{Z}_4 \times \mathbb{Z}_2$), so Let $i^2 = z_1$, $j^2 = z_2$. We note that $ij \in G_3$, so $|ij| = 4$ thus let $z_3 = (ij)^2$. This tells us that $z_3 = (ij)^2 = ijij = iz'ij^2 = z'i^2z_2 = z'z_1z_2$. by multiplying both sides by $z_3z'$, we get $z' = z_1z_2z_3$. Since $z' \neq e$ (see theorem 4.6) we can use the negation of theorem 4.1.3.1 (if $abc \neq e$, then at least two of the elements are equal), to state two possibilities (a) all three of $z_1, z_2, z_3$ are equal, or (b) two are equal.

**Assume all three of $z_1, z_2, z_3$ are equal**

In this case, we note that $z' = z_1z_2z_3 = z_1^3 = z_1$, so form the set $H = \{e, i, z_1, z_1i = i^{-1}, j, z_1j = j^{-1}, ij, z_1ij = ji = (ij)^{-1}\}$. This is a subgroup with the inverse shown (as $ij(ji) = ij^2i = iz_1i = ii^2i = i^4 = e$), and the subgroup is closed with the commuting rule. Next, grab a $z \neq z_1$, and form the group $\{e, z\} = \langle z\rangle = K$. We note that $H \cap K = \{e\}$, and $|HK| = |H||K|/|H \cap K|$=8*2/1=16, so $HK = G$. Also, since K is a subgroup of the center, $hk = kh$ for all $h \in$ H, $k \in$ K. Then G is the direct internal product of $H \cong \mathcal{Q}$ and $K \cong \mathbb{Z}_2$, so $G \cong H \times K \cong \mathcal{Q} \times \mathbb{Z}_2$. This group has representation $G = \{i^\alpha j^\beta z^\gamma : i^2 = j^2, i^4 = j^4 = z^2 = e, ji = i^{-1}j, zi = iz, jz = zj\}$.

**Assume two of $z_1, z_2, z_3$ are equal**

Without loss of generality, assume $z_2 = z_3$ (we can do that, as in this case $z' = z_1z_2z_3 = z_1$, versus when $z_1 = z_2$, then $z' = z_1z_2z_3 = z_3$ or $z_1 = z_3$, in which case $z' = z_1z_2z_3 = z_1z_3z_2 = z_2$, so no matter which two of $i, j$, or $ij$ have their squares equal, we always get the third as the commutator). In this case, $\langle i\rangle = \{e, i, z_1, z_1i\}$, and $\langle j\rangle = \{e, j, z_2, z_2j\}$. For these two subgroups $\langle i\rangle \cap \langle j\rangle = \{e\}$, and $|\langle i\rangle||\langle j\rangle| = |\langle i\rangle|\langle j\rangle|/|\langle i\rangle \cap \langle j\rangle| = 4 * 4/1 = 16$, so we have sixteen product of the form $i^\alpha j^\beta$, and a commuting rule of $ji = z'ij = i^2ij = i^{-1}j$. Thus this group has a representation $G = \{i^\alpha j^\beta : i^4 = j^4 = e, ji = i^{-1}j\}$. this is called the semidirect product of $\mathbb{Z}_4$ and $\mathbb{Z}_4$, $G \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_4$. We now have all groups with $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, shown in the table below:

| Name | Symbol | Representation | Center |
|---|---|---|---|
| Direct Product of $D_4$ and $\mathbb{Z}_2$ | $D_4 \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta c^\gamma : a^4 = b^2 = c^2 = e, ba = a^{-1}b, ca = ac, cb = bc\}$ | $\{e, a^2, c, a^2c\}$ |
| Semidirect product of Klein Group and $\mathbb{Z}_4$ | $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$ | $\{a^\alpha b^\beta : a^4 = b^4 = e, ba = a^{-1}b^{-1}\}$ | $\{e, a^2, b^2, a^2b^2\}$ |
| Direct Product of $\mathcal{Q}$ and $Z_2$ | $\mathcal{Q} \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta c^\gamma : a^2 = b^2, a^4 = b^4 = c^2 = e, ba = a^{-1}b, ca = ac, cb = bc\}$ | $\{e, a^2, c, a^2c\}$ |
| Semidirect product of $\mathbb{Z}_4$ and $\mathbb{Z}_4$ | $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ | $\{a^\alpha b^\beta : a^4 = b^4 = e, ba = a^{-1}b, \}$ | $\{e, a^2, b^2, a^2b^2\}$ |

Table 3: Non-ablian Groups of Order 16 with $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

## 5.2 $Z(G) \cong \mathbb{Z}_4$

Since $Z(G) \cong \mathbb{Z}_4$, we know that each $G_i$ contains a subgroup isomorphic to $\mathbb{Z}_4$. Since all the $G_i$ are abelian, and we know that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has no subgroup isomorphic to $\mathbb{Z}_4$, so the only possible isomorphism classes are $\mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$. We now have four possibilities, (5.2.1) all three $G_i \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ (5.2.2) two $G_i \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, (5.2.3) one $G_i \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, and (5.2.4) no $G_i \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

### 5.2.1 $G_1, G_2, G_3 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$

Grab $g_1, g_2$ with conditions [1], and let their orders be $|g_1| = |g_2| = 2$. We note that $g_1g_2 \in G_3$ is not in the center, so it has order 4 or 2. If we take $(g_1g_2)^2 = z'g_1^2g_2^2 = z'$. If $z' \in Z(G) \cong \mathbb{Z}_4$, has order 4, then $|g_ig_j| = 8$, which would imply $G_3 \cong \mathbb{Z}_8$ (not this case) so $|g_ig_j| = 4$, and $|z'| = 2$. If we grab a generator $z \in Z(G)$, (so $z^2 = z'$ and $\langle z\rangle = Z(G)$) then we see from theorem 3.4 that $G_1 = Z(G) \cup g_1Z(G)$, so every element in $G_1$ looks like $z^\alpha g_1^\beta$ (where since $|g_1| = 2$ means that $\beta = 0$ (it is in $Z(G)$) or $\beta = 1$ (it is in $g_1Z(G)$)). We next form $\langle g_2\rangle = \{e, g_2\}$, and note $G_1 \cap \langle g_2\rangle = \{e\}$. Thus the group $G_1\langle g_2\rangle$ has order $|G_1\langle g_2\rangle| = |G_1||\langle g_2\rangle|/|G_1 \cap \langle g_2\rangle| = 2 * 8/1 = 16$, so we have sixteen products of the form $z^\alpha g_1^\beta g_2^\gamma$, and a commuting rule $g_2g_1 = z'g_1g_2 = z^2g_1g_2$ so using this we can write a representation for $G$ as $G = \{z^\alpha g_1^\beta g_2^\gamma : z^4 = g_1^2 = g_2^2 = e, g_1z = zg_1, g_2z = zg_2g_2g_1 = z^2g_1g_2, \}$ (this is the group of Pauli matrices).

### 5.2.2 $G_1, \cong \mathbb{Z}_8,\ G_2, G_3 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$

As before, choose $g_2, g_3$ with conditions [1], and let them both be of order 2. We note that, as before $g_2 g_3 \in G_1$ is not in the center, but now $g_2 g_3$ has order 8. This means that $(g_1 g_2)^2 = z'$ (from above) has order 4. We note with the following theorem that this is impossible:

**Theorem 5.2.2.1** *If the commutator between any two elements is in the center, then it must be an element of order 2 (or e).*

Proof: we note that, regardless of whether the $G_i$'s are $\cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\cong \mathbb{Z}_8$ when we square any element, we get an element of $Z(G)$ (as squaring an element in $\mathbb{Z}_8$ puts you in the only subgroup of order 4, which in this case is the center, and squaring an element in $\mathbb{Z}_4 \times \mathbb{Z}_2$ either gives $(0,0)$ or $(2,0)$, both of which are in all three subgroups of order 4). If we have conditions [1] with $g_i, g_j$, and taking $g_j g_i^2 = g_i^2 g_j$. However, using our commutating rule, we get $g_j g_i^2 = z' g_i g_j g_i = z' g_i z' g_i g_j = z'^2 g_i^2 g_j$ and thus $g_i^2 g_j = z'^2 g_i^2 g_j$, and canceling the $g_i^2 g_j$ gives $e = z'^2$, so $|z'| = 2$ or $z' = e$.

We then get a contradiction if the commutator has order 4, so no group of order 16 has this property.

### 5.2.3 $G_1, G_2 \cong \mathbb{Z}_8, G_3 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$

Choose $g_1, g_2$ with conditions [1]. Since these are not in the subgroup of $G_2 \cong \mathbb{Z}_8$, of order 4 (which is $Z(G)$) they have order 8. Call $g_1^2 = z$, so $z \in Z(G)$ and $|z| = 4$. We note that we can choose a $g_2 \in G_2$ so ($|g_2| = 8$) and $g_2^2 = z$ (you can think of $g_1, g_2$ as corresponding to $1 \mod 8$, in each group and since their intersection is $Z(G)$, squaring them will give $2 \mod 8$). If we call $g_3 = g_1 g_2$, and let $z'$ be the commutator of $g_2$ and $g_1$, then $g_3 \in G_3$ (theorem 3.5) so $|g_3|$ is 4 or 2. We note that $g_3^2 = (g_1 g_2)^2 = z' g_1^2 g_2^2 = z' z^2$. We note that if the commutator is of order 4, then $g_3^2 = z$ or $g_3^2 = z^3$ (depending upon if $z' = z$ or $z' = z^{-1}$) either way, this would mean that $|g_3^2| = 4$ ($z$ and $z^3$ have orders 4), and thus $|g_1| = 8$ (as opposed to assuming the order of the commutator is 2, I want to deduce this from the properties of the group, so if I deduce that the commutator is of order 4, I get a contradiction). This would mean $G_1 \cong \mathbb{Z}_8$, which is not the case, hence $z'$ is of order 2, and from the properties of $Z(G) \cong \mathbb{Z}_4$, $z' = z^2$. This means that $g_3^2 = z' z^2 = z^4 = e$, so $|g_3| = 2$ and thus $\langle g_3 \rangle = \{e, g_3\}$ is a subgroup of order 2. and we note that, since $\langle g_1 \rangle = G_1$ that $G_1 \langle g_3 \rangle$ has order $|G_1 \langle g_3 \rangle| = |G_1|||\langle g_3 \rangle|/|G_1 \cap \langle g_3 \rangle| = 8 * 2/1 = 16$, so we have sixteen products of the form $g_1^\alpha g_2^\beta$, and since $g_3 g_1 = (g_1 g_2) g_1 = g_1 (g_2 g_1) = g_1 (z' g_1 g_2) = z' g_1 (g_1 g_2) = z' g_1 g_3$, we have a commuting rule $g_3 g_1 = z' g_1 g_3 = z^2 g_1 g_3 = g_1^4 g_1 g_3 = g_1^5 g_3$. Our group now has the representation $G = \{g_1^\alpha g_3^\beta : g_1^8 = g_3^2 = e, g_3 g_1 = g_1^5 g_3\}$ (this is the Isanowa or Modular Group of order 16).

### 5.2.4 $G_1, G_2, G_3 \cong \mathbb{Z}_8$

As before, choose $g_2, g_3$ with conditions [1], and with $|g_2| = 8 = |g_3|$ and picking $g_2$, $g_3$ such that $g_2^2 = z = g_3^2$. Again $g_1 = g_2 g_3$ with $g_1 \in G_1$, (so $|g_1| = 8$), and $g_1^2 = (g_2 g_3)^2 = z' z^2$. Note that if $z'$ is of order 2, the $|g_1| = 2$, which contradicts the order, so $|z'| = 4$. But this contradicts theorem 4.2.1, so there are no groups with this property.

We have hence considered all cases for $Z(G) \cong \mathbb{Z}_4$, and thus all cases of $|Z(G)| = 4$. We have listed the groups of order 16 with $Z(G) \cong \mathbb{Z}_4$ in the table below.

| name | symbol | representation | Center |
|---|---|---|---|
| Group of the Pauli Matrices | $SU(2)$ | $\{a^\alpha b^\beta c^\gamma : a^4 = b^2 = c^2 = e, ba = ab, ca = ac, cb = a^2bc\}$ | $\{e, a, a^2, a^3\}$ |
| Modular or Isanowa group of order 16 | $M_{16}$ | $\{a^\alpha b^\beta : a^8 = b^2 = e, ba = a^5 b\}$ | $\{e, a^2, a^4, a^6\}$ |

Table 4: Non-abelian groups of order 16 with centers $\mathbb{Z}_4$

# 6 $|Z(G)| = 2$

When the center is of order 2 (so $Z(G) \cong \mathbb{Z}_2$) then the central factor group has order $|G/Z(G)| = |G|/|Z(G)| = 16/2 = 8$, so $G/Z(G)$ is isomorphic to one of the groups of order 8, so we have 5 cases (a) $\mathbb{Z}_8$, (b) $\mathbb{Z}_4 \times \mathbb{Z}_2$, (c) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, (d) $D_4$, or (e) $\mathcal{Q}$.

(a) **assume** $G/Z(G) \cong \mathbb{Z}_8$

In this case, the central factor group is cyclic, so this implies that the group is abelian (theorem 3.1) which implies $Z(G) = G$ so $|Z(G)| = 16$, which contradicts that $|Z(G)| = 2$. Hence $G/Z(G) \ncong \mathbb{Z}_8$.

(b) **assume** $G/Z(G) \cong \mathbb{Z}_4 \times \mathbb{Z}_2$

We note that $\mathbb{Z}_4 \times \mathbb{Z}_2$ has three subgroups, two subgroups isomorphic to $\mathbb{Z}_4$ (which has 1 subgroup $\cong \mathbb{Z}_2$) and one subgroup $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (which has 3 subgroups $\cong \mathbb{Z}_2$). By the correspondence theorem, this means there are 3 subgroups of order 8 that contain $Z(G)$, two of which have one subgroup of order 4 that contain the center (call these $G_1, G_2$), and one subgroup of order 8 that contains three subgroups of order 4 with the center (call this $G_3$). We then prove the following theorem:

**Theorem 6.1** *If $|Z(G)| = 2$ and $G_i$ contains 1 subgroup of order 4 with the center, then $G_i$ is either isomorphic to $\mathbb{Z}_8$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$.*

Proof: We know from Theorem 3.1 that the center of $G_i$ contains $Z(G)$. However, from the properties of the groups of order 8, we see that, for the non-abelian groups ($D_4$ and $\mathcal{Q}$) they have three subgroups that contain the center, so groups with this property are not abelian. Likewise, they cannot be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, because every subgroup of order 2 is contained in three subgroups of order 4, and not one. Furthermore, if $G_i \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ then it must be in the subgroup $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, as the two subgroups $\cong \mathbb{Z}_4$ share their subgroup of order 2. The only two options are consequently $G_i \cong \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2$.

Therefore, since $G_1, G_2$ has 1 subgroup of order 4 containing the center, they are abelian. We now prove that this leads to a contradiction:

**Theorem 6.2** *If $G$ has two $G_i$ that are abelian (could also be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$), then the center has order at least 4.*

Proof: Let $G_i, G_j$ be distinct and abelian. We note that $G_iG_j$ is equal to the group (as $|G_iG_j| = |G_i||G_j|/|G_i \cap G_j| = 8 * 8/4 = 16$) and we note that, since they are abelian, every element in $G_i \cap G_j$ commutes with the elements in $G_i$ and $G_j$. Therefore, if $g \in G_i \cap G_j$, then $g$ commutes with every element in $G_1G_2$ ($g$ commutes with all $g_1 \in G_1$, $g_2 \in G_2$, and an arbitrary element in $G_1G_2$ is $g_1g_2$, thus $g(g_1g_2) = (gg_1)g_2 = g_1gg_2 = (g_1g_2)g)$. Since it commutes with every element in the group, it is in the center. This means that every element is $G_i \cap G_j$ is in the center, and since there are four elements in $G_i \cap G_j$, there must be four elements in the center.

Since we have two abelian groups $(G_1, G_2)$, the center must be of at least order 4, but this would contradict that the center is of order 2. Hence $G/Z(G) \not\cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

(c) **assume** $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

To begin, we note that since every element in the factor group has order 2, $Z(G) = (gZ(G))^2 = g^2Z(G)$, so every element squared is in the center. We note that since the group is not abelian, there exists 1 element $g$ that does not have $g^2 = e$ (theorem 4.1.1.1). Since $g^2 \in Z(G)$, however, we note that $g^2 = z$. We note that $\langle g \rangle = \{e, g, g^2 = z, g^3 = zg\}$ is a subgroup containing the center, so by the correspondence theorem there is a factor group of order 2 in the center (in this case, $\langle gZ(G) \rangle = \{Z(G), gZ(G)\}$). We note that this group is contained in the three subgroups of order 4 (each of the the subgroup of order 2 is contained in 3 subgroups in $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$). If we call these $G_1, G_2, G_3$, then we note at least two of these are nonabelian (if two are abelian it implies a center of order 4, see Theorem 5.2). Without loss of generality assume $G_2, G_3$ are nonabelian. Then we note that if we let $g_2 \in G_2, g_3 \in G_3$ which are not in their intersection (which in this case is $\langle g \rangle$). then we note that $g_2g_3 \in G_1$ and is not in $\langle g \rangle$. then we note that, since $g \notin \mathrm{Z(G_2)}$, $\mathrm{Z(G_3)}$ (which are of order 2 so equal the center) when we commute we pick up a z, $g_2g = zgg_2$ (same for $g_3$). If we then take $(g_2g_3)g$, we see that $(g_2g_3)g = g_2(g_3g) = g_2(zgg_3) = z(g_2g)g_3 = z(zgg_2)g_3 = z^2g(g_2g_3) = g(g_2g_3)$, so $g$ commutes with $g_2g_3 \in G_1$, and we note that $g$ commutes with $\langle g \rangle \cup (g_2g_3)\langle g \rangle$, which are eight elements $G_1$ hence it equals $g_1$, hence $g_1 \in \mathrm{Z}(G_1)$, and now the center of $G_1$ has at least three elements $(g, e, z)$ it has to be abelian. This center has no elements of order 8, as these do not have $g^2 = z$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has no elements of order 2, so $G_1 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

We next let $g_1 \in G_1, g_1 \notin \langle g \rangle$. If $h_2 \in G_2$ commutes with $g_1$, then $g_1^\alpha h_2^\beta g^\gamma$ is an abelian group with eight elements, as everything commutes with everything else, and has eight elements by two options for each greek letter (if either an element of order 8, you can use that $g_1^2$ and/or $h_2^2$ is equal to $z$) then $G_i = \{g_1^\alpha h_2^\beta z^\gamma\}$ is an abelian group, we have two abelian groups of order 8, which implies that $|Z(G)| \geq 4$ which contradicts that $|Z(G)| = 2$. Thus $h_2$ cannot commute with any elements in $G_1$ that are not in $\langle g \rangle$. Thus $g_2$ does not commute with $g_1$, but we note that $g_1g$ is not in $\langle g \rangle$ (as then $g_1 = g^{-1}g^n = g^{n-1}$ is in $\langle g \rangle$). But $gg_1g_2 = g(zg_2g_1) = z(gg_2)g_1 = z^2g_2gg_1 = g_2(gg_1)$ has $g_2$ commuting with $gg_1$, which would contradict the fact that it does not commute, which if it did commute we could construct a second abelian group and thus $|Z(G)| \geq 4$, which contradicts that $|Z(G)| = 2$.

(d) **assume** $G/Z(G) \cong \mathcal{Q}$

We note from the properties of $\mathcal{Q}$ that the factor group has three subgroups of order 4, each with 1 subgroup of order 2. By the Correspondence Theorem, this means there are 3 subgroups of order 8 in $G$, each with 1 subgroup of order 4 (call these $G_i$). By Theorem 6.1, all three of these are abelian, and since there are 2 abelian $G_i$, the center must have order 4, contradicting that the center has order 2. Hence no groups of order 16 have this property.

(e) **assume** $G/Z(G) \cong D_4$

In this case we note that the factor group has 1 subgroup of order 4 with 1 subgroup of order 2, and 2 subgroups of order 4 with three subgroups of order 2. By the Correspondence Theorem, we have 1 subgroup of order 8 with 1 subgroup of order 4 containing the center (call this $G_1$ and from Theorem 5.1, $G_1 \cong \mathbb{Z}_8$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$), and two subgroups of order 8 with three subgroups containing the center (denote these $G_2$, $G_3$). We note that both $G_2, G_3$ are nonabelian, as if they were abelian then by theorem 5.2, $Z(G)$ would have order 4. Before we determine the nature of these subgroups we need to determine how elements commute. This means finding out the commutator subgroup:

**Definition** *the commutator subgroup, $G'$, is defined as $G' = \langle ghg^{-1}h^{-1} : g, h \in G\rangle$ (Judson, 202).*

Note that $Z(G) \subset \mathrm{Z(G_2)}$, and in these nonabelian groups of order 8, the center is of order 2, hence $Z(G) = \mathrm{Z(G_2)}$. Also, based on the properties of these groups, the center is contained in the commutator (for $G_2 \cong D_4$, then $z = r^2$ is the nontrivial element of the center, and $z = r^2 = srs^{-1}r^{-1}$, so $z$ an element of $G'$. for $G_2 \cong \mathcal{Q}$, then $z = -1 = i^2$ is a nontrivial element of the center, and $z = -1 = i^2 = iji^{-1}j^{-1}$ is in $G'$). Let $Z(G), g'Z(G)$ be the center of $G/Z(G)$ (so $g'$ is equivalent to $r^2$ in $D_4$). If we pick $g_1Z(G), g_2Z(G)$ in the factor group so they do not commute, then from the properties of $D_4$, they pick up an element of the center when they commute $g_1g_2Z(G) = (g_1Z(G))(g_2Z(G)) = (g'Z(G))(g_2Z(G))(g_1Z(G)) = g'g_1g_2Z(G)$, so $g_1g_2 = g'z_0g_2g_1$, for some $z_0$ in the center, hence $g', zg'$ are in the commutator subgroup (as regardless of $z_0$ the fact that the commutator is a subgroup means we can multiply by its inverse in the center (which is in the commutator) and get $g'$, and from this $zg'$). Note that these four elements form a subgroup, and we have no other generators for $G'$ (assume $gh = chg$ with c in the commutator. Then $gZ(G)hZ(G) = ghZ(G) = chgZ(G) = cZ(G)hZ(G)gZ(G)$ and in the factor group, we either pick up a $g'Z(G) = \{g', zg'$ or $Z(G) = \{e, z\}$, so commuting picks up one of these four elements, which are the four elements in the center). This we get no more generators, thus $G' = \{e, z, g', zg'\}$ is the commutator. We note that $G'$ corresponds to the center of the factor group, and the center is contained in the four subgroups of order 4, hence by the correspondence theorem this subgroup is in all three subgroups of order 8 containing the center ($G' \subset G_1, G_2, G_3$), and note that $|G_i \cap G_j| = 4$, so the intersection of two $G_i$ is $G'$.
We now need to determine the isomorphism classes of $G_1, G_2G_3$. We will determine the structure of $G_1$ (the abelian one).

**Theorem 6.3** *if $G/Z(G) \cong D_4$ then one of the $G_i$ is cyclic*

Proof: We already know that $G_1 \cong \mathbb{Z}_8$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$, so we will do a proof by contradiction. Assume $G_1 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Note from theorem 6.1 that this means that the one subgroup of order 4 containing the center is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. We know $G_2, G_3$ are nonabelian, so they are isomorphic to $\mathcal{Q}$ or $D_4$. However, neither are isomorphic $\mathcal{Q}$, as the only subgroups of order 4 in $\mathcal{Q}$ are cyclic, so this would mean that $G' = G_1 \cap G_i \cong \mathbb{Z}_4$ is cyclic, which contradicts that it is $\mathbb{Z}_2 \times \mathbb{Z}_2$. This leaves both $G_2, G_3 \cong D_4$. since there are five elements of order 2 in $D_4$, we can pick $g_2$ such that $|g_2| = 2$. We note that, for $g_1 \in G_1$, (and commutator $g'$ between $g_2$ and $g_1$). Then $g_1 = g_2^2g_1 = g_2g'g_1g_2 = zg'g_2g_1g_2 = zg'g'g_1g_2^2 = zg'^2g_1$ (we note that $g', g_2 \in G_2$ commute with an element of center from the properties of groups of order 8). Canceling out $g_1$ gives $e = g'^2z$ and thus $g'^2 = z^{-1} = z$ (as inverse of element of order 2 is itself) thus $|g_{\prime}| = 4$. But this contradicts that the group of order 4 containing the center is $\cong \mathbb{Z}_4 \times \mathbb{Z}_2$, hence this is a contradiction, so there is no way for $G_1 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

Now that we know $G_1 \cong \mathbb{Z}_8$, we have three scenarios, (5.0.1) $G_2, G_3 \cong \mathcal{Q}$, (5.0.2) $G_2 \cong \mathcal{Q}, G_3 \cong D_4$, or (5.0.3) $G_2, G_3 \cong D_4$.

### 5.0.1 $G_2, G_3 \cong \mathcal{Q}$

Grab $g_1, g_2$ with conditions [1]. Note that, since $G_1 \cong \mathbb{Z}_8$ shares a subgroup of order 4 with $G_2 \cong \mathcal{Q}$, we have ($g_1^4 = g'^2 = g_2^2 = z$). Also, $g_1g_2 \in G_3$, so $z = (g_1g_2)^2 = g_1g_2g_1g_2 = g_1g'g_1g_2g_2 = g'g_1^2z$, so $e = g'g_1^2$, so $g' = (g_1^2)^{-1} = g_1^{-2} = g_1^6$. If we take $\langle g_1\rangle = G_1$, and $\langle g_2\rangle = \{e, g_2, z, zg_2\}$ (so $G_1 \cap \langle g_2\rangle = \{e, z\}$), and form $G_1\langle g_2\rangle$, then this has order $|G_1\langle g_2\rangle| = ||G_2||\langle g_2\rangle|/|G_1 \cap \langle g_2\rangle| = 8 * 4/2 = 16$, so there are 16 elements of the form $g_1^\alpha g_2^\beta$ and commuting rule $g_2g_1 = g'g_1g_2 = g_1^6g_1g_2 = g_1^7g_2 = g_1^{-1}g_2$. So our group has the representation $G = \{g_1^\alpha g_2^\beta : g_1^4 = g_2^2, g_1^8 = g_2^4 = e, g_2g_1 = g_1^{-1}g_2\}$, This is the generalized quaternions, (also dicyclic group of degree 4, $Dic_4$).

### 5.0.2 $G_2 \cong D_4, G_3 \cong \mathcal{Q}$

Grab $g_1, g_3$ with conditions [1], and note that $g_1g_3 \in G_2 \cong D_4$ is not in the cyclic subgroup of order 4, so has order 2, $(g_1g_3)^2 = e$. This means that $e = (g_1g_3)^2 = g_1g_3g_1g_3 = g_1g'g_1g_3^2 = g_1^2g'(g_1^4) = g'g_1^6$, so $g' = (g_1^6)^{-1} = g_1^{-6} = g_1^2$ If we let $g_2 = g_1g_3$, then to determine commutator between $g_2$ and $g_1$, note $g_2g_1 = g_1g_3g_1 = g_1g'g_1g_3 = g_1^2g_1(g_1g_3) = g_1^3g_2$, and we note that since $|g_2| = 2$, $\langle g_2\rangle = \{e, g_2\}$, and $\langle g_1\rangle = G_1$, and $|G_1\langle g_2\rangle| = |G_1||\langle g_2\rangle|/|G_1 \cap \langle g_2\rangle| = 8 * 2/1 = 16$, so sixteen elements of the form $g_1^\alpha g_2^\beta$, so using the orders and commuting rules we get a group representation of $G = \{g_1^\alpha g_2^\beta : g_1^8 = g_2^2 = e, g_2g_1 = g_1^3g_2\}$. This is the semidihedral group of degree 2, $SD_2$.

### 5.0.3 $G_2, G_3 \cong D_4$

Grab $g_1 \in G_1$, $g_2 \in G_2$ with conditions [2]. Let $g'$ be the commutator between them $g_2g_1 = g'g_1g_2$, and $|g_2| = 2$. Since $g_1g_2 \in G_3 \cong D_4$, and not in cyclic group of order 4, $|g_1g_2| = 2$. We note that $e = (g_1g_2)^2 = g_1g_2g_1g_2 = g_1g'g_1g_2^2 = g'g_1^2$, so $g' = (g_1^2)^{-1} = g_1^6$ Next, we note that $\langle g_1\rangle = G_1$, and $\langle g_2\rangle = \{e, g_2\}$, so $|G_1\langle g_2\rangle| = |G_1||\langle g_2\rangle|/|G_1 \cap \langle g_2\rangle| = 8 * 2/1 = 16$, so sixteen elements of the form $g_1^\alpha g_2^\beta$ with commuting rule $g_2g_1 = g'g_1g_2 = g_1^6g_1g_2 = g_1^7g_2 = g_1^{-1}g_2$. This gives a representation of $G = \{g_1^\alpha g_2^\beta : g_1^8 = g_2^2 = e, g_2g_1 = g_1^{-1}g_2\}$ (this is the dihedral group of degree 8, $\mathrm{D}_8$.

Thus we have determined the 14 groups of order 16, which are all collected in a table in the appendix

| name | symbol | representation | Center |
|---|---|---|---|
| Dicyclic Group of Degree 4 | $\mathrm{Dic}_4$ | $\{a^\alpha b^\beta : a^4 = b^2a^8 = b^4 = e, ba = a^{-1}b\}$ | $\{e, a^4\}$ |
| Semidihedral group of degree 2 | $\mathrm{SD}_2$ | $\{a^\alpha b^\beta : a^8 = b^2 = e, ba = a^3b\}$ | $\{e, a^4\}$ |
| Dihedral group of degree 8 | $\mathrm{D}_8$ | $\{a^\alpha b^\beta : a^8 = b^2 = e, ba = a^{-1}b\}$ | $\{e, a^4\}$ |

Table 5: Groups of order 16 with $Z(G) \cong \mathbb{Z}_2$

Thus we have now determined all 14 groups of order 16, which are collected in the appendix.

# 7 Resources

## References

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Appendix 1: The representations of the groups of order 16

| name | symbol | representation | Center |
|---|---|---|---|
| Integers mod 16 | $\mathbb{Z}_{16}$ | $\{a^\alpha : a^{16} = e\}$ | Abelian |
| Direct Product of $\mathbb{Z}_8$ and $\mathbb{Z}_2$ | $\mathbb{Z}_8 \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta : a^8 = b^2 = e, ba = ab\}$ | Abelian |
| Direct Product of $\mathbb{Z}_4$ and $\mathbb{Z}_4$ | $\mathbb{Z}_4 \times \mathbb{Z}_4$ | $\{a^\alpha b^\beta : a^4 = b^4 = e, ba = ab\}$ | Abelian |
| Direct Product of $\mathbb{Z}_4$, $\mathbb{Z}_2$ and $\mathbb{Z}_2$ | $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta c^\gamma : a^4 = b^2 = c^2 = e, ba = ab, ca = ac, cb = bc\}$ | Abelian |
| Direct product of $\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_2$ and $\mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta c^\gamma d^\delta : a^2 = b^2 = c^2 = d^2 = e, ba = ab, ca = ac, da = ad, cb = bc, db = bd, dc = cd\}$ | Abelian |
| Direct Product of $D_4$ and $\mathbb{Z}_2$ | $D_4 \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta c^\gamma : a^4 = b^2 = c^2 = e, ba = a^{-1}b, ca = ac, cb = bc\}$ | $\{e, a^2, c, a^2c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ |
| Semidirect product of Klein Group and $\mathbb{Z}_4$ | $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$ | $\{a^\alpha b^\beta : a^4 = b^4 = e, ba = a^{-1}b^{-1}\}$ | $\{e, a^2, b^2, a^2b^2\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ |
| Direct Product of $\mathcal{Q}$ and $\mathbb{Z}_2$ | $\mathcal{Q} \times \mathbb{Z}_2$ | $\{a^\alpha b^\beta c^\gamma : a^2 = b^2, a^4 = b^4 = c^2 = e, ba = a^{-1}b, ca = ac, cb = bc\}$ | $\{e, a^2, c, a^2c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ |
| Semidirect product of $\mathbb{Z}_4$ and $\mathbb{Z}_4$ | $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ | $\{a^\alpha b^\beta : a^4 = b^4 = e, ba = a^{-1}b, \}$ | $\{e, a^2, b^2, a^2b^2\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ |
| Group of the Pauli Matrices | $SU(2)$ | $\{a^\alpha b^\beta c^\gamma : a^4 = b^2 = c^2 = e, ba = ab, ca = ac, cb = a^2bc\}$ | $\{e, a, a^2, a^3\} \cong \mathbb{Z}_4$ |
| Modular or Isanowa group of order 16 | $M_{16}$ | $\{a^\alpha b^\beta : a^8 = b^2 = e, ba = a^5b\}$ | $\{e, a^2, a^4, a^6\} \cong \mathbb{Z}_4$ |
| Dicyclic Group of Degree 4 | $\mathrm{Dic}_4$ | $\{a^\alpha b^\beta : a^4 = b^2a^8 = b^4 = e, ba = a^{-1}b\}$ | $\{e, a^4\} \cong \mathbb{Z}_2$ |
| Semidihedral group of degree 2 | $\mathrm{SD}_2$ | $\{a^\alpha b^\beta : a^8 = b^2 = e, ba = a^3b\}$ | $\{e, a^4\} \cong \mathbb{Z}_2$ |
| Dihedral group of degree 8 | $\mathrm{D}_8$ | $\{a^\alpha b^\beta : a^8 = b^2 = e, ba = a^{-1}b\}$ | $\{e, a^4\} \cong \mathbb{Z}_2$ |