Propositional Logics and their Algebraic Equivalents

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1 Introduction

This paper is meant as an introduction to the study of logic for undergraduate mathematicians having completed a year-long course in abstract algebra. It is possible to investigate a logic as an algebraic structure, the properties of that structure giving insight in to the logic itself. We will discuss that connection between Boolean algebras and the system of classical propositional logic. We will also introduce the idea of different logical systems, and investigate their algebraic equivalents as well.

2 Formal Logic Systems

Before we can discuss the algebraic representations of formal logics, we must first define the semantics and syntax of a formal logic system.

Definition 2.1. A formal logic system S is comprised of the following objects:

1. A language L, which is generated by:
   (a) A finite set of symbols; sequences of these symbols are called *words.*
   (b) A set of formulas, which are special cases of words.
   (c) A set of rules for creating formulas called the *syntax* of L.
2. A deductive system H, which consists of:
   (a) A set of formulas called the *axioms* of H.
   (b) A set of rules called the rules of inference on S. These rules along with the axioms are used to generate new formulas, which are called the *theorems* of S.

Definition 2.2. A *propositional language* L is a language whose symbols (besides parentheses) belong to one of two sets: the set of atomic formulas or the set of connectives. The atomic formulas are the variables and constants in the language of L. A *connective* is a symbol $*$ that acts as an unary or binary function on other formulas. In algebraic parlance, a connective is an operator on the formulas of L.

The formulas of L are defined inductively as follows:

1. Every variable is a formula, and every constant is a formula.
2. If $*$ is an unary connective, then for any formula ϕ, $*\phi$ is also a formula.
3. If $*$ is a binary connective, then for any formulas ϕ, ψ, $\phi * \psi$ is also a formula.

We will include parentheses in our language to denote order of operations. Parentheses are added to formulas according to the rule: if ϕ is a formula, then (ϕ) is a formula.

Note. One could also view the constants of a propositional language as 0-ary connectives (i.e., connectives with no arguments).

Definition 2.3. A *propositional logic system* is a logic system over a propositional language.

2.1 Consequence Relations

If S is a formal logic system with a deductive system H and X is an arbitrary set of formulas, then we say that a formula ϕ is *deducible* from X if there is a finite sequence of formulas $\{\phi_1, \phi_2, \ldots, \phi_n\}$ such that $\phi = \phi_n$ and for every ϕ_i in the sequence, ϕ_i either belongs to X, is an axiom of H, or is obtained from the formulas $\{\phi_1, \ldots, \phi_{i-1}\}$ via one of the inference rules of H. Such a sequence is called a *proof* of ϕ with hypotheses X [3].

A *consequence relation* is a relation $\vdash$ between sets of formulas and formulas defined $X \vdash \phi$ if ϕ is deducible from X by the rules and axioms of its corresponding deductive system. Then $\vdash$ has the following properties:

1. if $\phi \in X$ then $X \vdash \phi$
2. if $X \vdash \phi$ and $X \subseteq Y$ then $Y \vdash \phi$
3. if $X \vdash \phi$ and for every $\psi \in X, Y \vdash \psi$, then $Y \vdash \phi$

For consequence relations where the set on the left contains a single formula, we will simply write $\phi \vdash \psi$.

Definition 2.4. The relation $\Leftrightarrow$ over the set of formulas F of L is defined $\phi \Leftrightarrow \psi$ if and only if (i) $\phi \vdash \psi$ and (ii) $\psi \vdash \phi$. Then $\Leftrightarrow$ is an equivalence relation on F; it is the same as the statement "ϕ if and only if ψ." For our purposes, we will write the equivalence class of the formula ϕ as $[\phi]$.

Definition 2.5. Any formula ϕ such that $\emptyset \vdash \phi$ is a *tautology*; it is always true regardless of premises. The axioms of a deductive system are all tautologies.

Theorem 2.1. *The relation $\vdash$ is a partial order on the equivalence classes of formulas.*

Proof. $\vdash$ satisfies the three properties of a partial ordering:

1. Reflexivity: $\phi \in [\phi]$, therefore $\phi \vdash \phi$.
2. Transitivity: From $\phi \vdash \psi$, we know that there is a list of formulas proving ψ from ϕ and the axioms of H, and from $\psi \vdash \chi$ we know that there is another list of formulas proving χ from ψ. Concatenating these lists together (with some caveats: see [1]) produces a proof of χ from ϕ. So $\phi \vdash \chi$.
3. Anti-symmetry: if $\phi \vdash \psi$ and $\psi \vdash \phi$ then $[\phi] = [\psi]$.

$\square$

3 Propositional Logic Systems

The language of Classical Propositional Logic (CPL) contains the variables $\{p, q, \dots\}$ (denoting individual propositions), the constants $\{\top, \bot\}$ (denoting true and false), the unary connective $\{\neg\}$ (denoting negation), and the binary connectives $\{\wedge, \vee, \rightarrow\}$ (denoting conjunction, disjunction, and implication).

3.1 The axiomatic approach to proof

However, it is also possible to provide a set of axioms, definitions, and rules of inference which describe H. There are many possible axiomatizations for CPL; we will use a comparatively large set of eleven axioms in our formulation of CPL [1], which are given in table 1.

Table 1: 11 Axioms for CPL

Name	Axiom schema (ϕ, ψ, χ formulas)
THEN-1	$\phi \rightarrow (\chi \rightarrow \phi)$
THEN-2	$(\phi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\phi \rightarrow \chi) \rightarrow (\phi \rightarrow \psi))$
AND-1	$\phi \wedge \chi \rightarrow \phi$
AND-2	$\phi \wedge \chi \rightarrow \chi$
AND-3	$\phi \rightarrow (\chi \rightarrow (\phi \wedge \chi))$
OR-1	$\phi \rightarrow \phi \vee \chi$
OR-2	$\chi \rightarrow \phi \vee \chi$
OR-3	$(\phi \rightarrow \psi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\phi \vee \chi \rightarrow \psi))$
NOT-1	$(\phi \rightarrow \chi) \rightarrow ((\phi \rightarrow \neg\chi) \rightarrow \neg\phi)$
NOT-2	$\phi \rightarrow (\neg\phi \rightarrow \chi)$
NOT-3	$\phi \vee \neg\phi$

We define the contradiction $\bot$ to be $\bot \Leftrightarrow \phi \wedge \neg\phi$ and the tautology $\top$ as $\emptyset \vdash \top$. The only rule of inference of CPL is *Modus Ponens*, which states:

$$\{\phi, \phi \rightarrow \psi\} \vdash \psi$$

That is, if we know that ϕ and $\phi \rightarrow \psi$ are true, we may infer ψ is true.

Theorem 3.1 (Deduction Theorem)**.** *If $\{\phi_1, \dots, \phi_n, \phi\} \vdash \psi$, then we can infer that $\{\phi_1, \dots, \phi_n\} \vdash \phi \rightarrow \psi$ (For proof, see [1, ch. 1]).*

Corollary 3.2. *If $\phi \vdash \psi$, then $\emptyset \vdash \phi \rightarrow \psi$.*

Definition 3.1. Intuitionistic Propositional Logic (IPL) is a logical system which is identical to CPL in every way except that it lacks the axiom NOT-3, $\phi \vee \neg\phi$, also known as the "law of the excluded middle."

4 The Algebraic Structure of Logic Systems

Here is an example of a basic proof in CPL:

Lemma 4.1. $\{\phi, \psi\} \vdash \phi \wedge \psi$.

Proof.

(1)	ϕ	hypothesis
(2)	ψ	hypothesis
(3)	$\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi))$	AND-3
(4)	$\psi \rightarrow (\phi \wedge \psi)$	Modus Ponens on 1, 3
(5)	$\phi \wedge \psi$	Modus Ponens on 2, 4

□

Theorem 4.2. *The equivalence classes of formulas(given by $\Leftrightarrow$) of both CPL and IPL form a lattice λ under $\wedge$ and $\vee$.*

Proof. The *least upper bound* of the formulas ϕ and ψ is $\phi \vee \psi$. That is, the following statements $\phi \vdash (\phi \vee \psi)$ and $\psi \vdash (\phi \vee \psi)$ are true, and for any formula χ for which $\phi \vdash \chi$ and $\psi \vdash \chi$, $(\phi \vee \psi) \vdash \chi$.

First, we show that $(\phi \vee \psi)$ is an upper bound:

$\phi \vdash (\phi \vee \psi)$:

(1)	ϕ	hypothesis
(2)	$\phi \rightarrow (\phi \vee \psi)$	OR-1
(3)	$\phi \vee \psi$	Modus Ponens on (1), (2)

$\psi \vdash (\phi \vee \psi)$:

(1)	ψ	hypothesis
(2)	$\psi \rightarrow (\phi \vee \psi)$	OR-2
(3)	$(\phi \vee \psi)$	Modus Ponens on (1), (2)

Now, we show that $(\phi \vee \psi)$ is a *least* upper bound. That is, if $\phi \vdash \chi$ and $\psi \vdash \chi$, then $(\phi \vee \psi) \vdash \chi$. This is equivalent (by Theorem 3.1) to the statement $\{\phi \rightarrow \chi, \psi \rightarrow \chi, \phi \vee \psi\} \vdash \chi$.

$\{\phi \to \chi, \psi \to \chi, \phi \vee \psi\} \vdash \chi$:

(1)	$\phi \to \chi$	hypothesis
(2)	$\psi \to \chi$	hypothesis
(3)	$\phi \vee \psi$	hypothesis
(4)	$(\phi \to \chi) \to ((\psi \to \chi) \to (\phi \vee \psi \to \chi))$	OR-3
(5)	$(\psi \to \chi) \to (\phi \vee \psi \to \chi)$	Modus Ponens on (1), (4)
(6)	$\phi \vee \psi \to \chi$	Modus Ponens on (2), (5)
(7)	χ	Modus Ponens on (3), (6)

Similarly, the *greatest lower bound* of the formulas ϕ and ψ is $\phi \wedge \psi$:

$(\phi \wedge \psi) \vdash \phi$:

(1)	$\phi \wedge \psi$	hypothesis
(2)	$(\phi \wedge \psi) \to \phi$	AND-1
(3)	ϕ	Modus Ponens on (1), (2)

$(\phi \wedge \psi) \vdash \psi$:

(1)	$\phi \wedge \psi$	hypothesis
(2)	$(\phi \wedge \psi) \to \psi$	AND-2
(3)	ψ	Modus Ponens on (1), (2)

$\{\chi \to \phi, \chi \to \psi, \chi\} \vdash \phi \wedge \psi$:

(1)	χ	hypothesis
(2)	$\chi \to \phi$	hypothesis
(3)	$\chi \to \psi$	hypothesis
(4)	ϕ	Modus Ponens on (1), (2)
(5)	ψ	Modus Ponens on (1), (3)
(6)	$\phi \wedge \psi$	Lemma 4.1 on (4), (5)

$\square$

The largest element of λ is the class of tautologies, $[\top]$. The smallest is the class of contradictions, $[\bot]$.

From the basic properties of lattices, we know that the commutative, associative, idempotent, and absorption laws hold for the logical connectives $\wedge$ and $\vee$ (See Theorem 19.2 in [4]). The following theorem will allow us to further characterize the algebraic structure of these logic systems.

Theorem 4.3. *In CPL and IPL, conjunction distributes over disjunction and vice-versa. This is called the* Rule of distribution*, and it is stated formally as follows:*

$$\phi \wedge (\psi \vee \chi) \Leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$$
$$\phi \vee (\psi \wedge \chi) \Leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi)$$

Proof. We will prove the forward direction for conjunction distributing over disjunction. We will prove two separate statements:

(i) $\phi \vdash (\phi \vee \psi) \wedge (\phi \vee \chi)$

(1)	ϕ	hypothesis
(2)	$\phi \rightarrow (\phi \vee \psi)$	OR-1
(3)	$\phi \rightarrow (\phi \vee \chi)$	OR-2
(4)	$\phi \vee \psi$	Modus Ponens on 1, 2
(5)	$\phi \vee \chi$	Modus Ponens on 1, 3
(6)	$(\phi \vee \psi) \wedge (\phi \vee \chi)$	Lemma 4.1 on 4, 5

(ii) $(\psi \wedge \chi) \vdash (\phi \vee \psi) \wedge (\phi \vee \chi)$

(1)	$\psi \wedge \chi$	hypothesis
(2)	$(\psi \wedge \chi) \rightarrow \psi$	AND-1
(3)	$(\psi \wedge \chi) \rightarrow \chi$	AND-2
(4)	ψ	Modus Ponens on 1, 2
(5)	χ	Modus Ponens on 1, 3
(6)	$\psi \rightarrow (\phi \vee \psi)$	OR-2
(7)	$\chi \rightarrow (\phi \vee \chi)$	OR-2
(8)	$\phi \vee \psi$	Modus Ponens on 4, 6
(9)	$\phi \vee \chi$	Modus Ponens on 5, 7
(10)	$(\phi \vee \psi) \wedge (\phi \vee \chi)$	Lemma 4.1 on 8, 9

Theorem 4.2 tells us that because $(\phi\wedge(\psi\vee\chi))$ is the least upper bound of ϕ and $(\psi\vee\chi)$, and $(\phi\vee\psi)\wedge(\phi\vee\chi)$ is also an upper bound, then $\phi\wedge(\psi\vee\chi) \vdash (\phi\vee\psi)\wedge(\phi\vee\chi)$.

For the proof of the converse, see Podnieks and Detlovs [1, ch. 2].

By the principle of duality [4, p. 302], we may infer that disjunction distributes over conjunction.

□

4.1 Boolean Algebra

We have shown that conjunction and disjunction on CPL and IPL follow the laws of commutativity, associativity, idempotence, absorption, and distribution. At this point, we have not shown that there is any difference between the algebraic models of each logic. The next step we take will expose a difference, as it will depend on the axiom NOT-3, which belongs only to CPL. The final step in characterizing the algebra of CPL as a Boolean algebra is to show that the the equivalence classes of its formulas are complemented.

Theorem 4.4. *The set of equivalence classes of formulas in CPL is a Boolean algebra over the operations of conjunction and disjunction.*

Proof. It only remains to show that CPL is complemented to establish this fact. For any formula ϕ in CPL, its complement is $\neg\phi$. We must show that $\phi \vee \neg\phi \vdash \top$ and $\phi \wedge \neg\phi \vdash \bot$. The former is trivial, as its hypothesis is the axiom NOT-3 which is tautological by definition. Similarly, the statement $\phi \wedge \neg\phi$ is the definition of a contradiction, both in CPL and in IPL. □

The connection of CPL to Boolean algebras is the reason we can use truth tables to evaluate formulas. Because propositions and formulas have only two truth-values in CPL and there are a finite number of terms in every formula, we can build a table of all the possible truth values of the terms in a formula to ascertain the validity of the formula itself. In the truth-table approach, the consequence relation is defined as follows: for a set of formulas F and a formula ϕ, $F \vdash \phi$ if and only if every assignment of truth values that assigns $\top$ to all $\psi \in F$ also assigns $\top$ to ϕ [3].

From the truth table, it is easy to see that $\{p, q\} \vdash (p \wedge q)$, because every truth valuation that assigns $\top$ to p and q also assigns $\top$ to $(p \wedge q)$. We can also see that the connective $\rightarrow$ is equivalent to the conjunction and disjunction operators by $p \rightarrow q \Leftrightarrow (\neg p \vee q)$.

4.2 Heyting Algebras

The system of intuitionistic propositional logic is not complemented because it lacks the axiom NOT-3. Furthermore, the implication connective $\rightarrow$ cannot be

Table 2: Truth table for propositions and connectives in CPL

p	q	$\top$	$\bot$	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$
$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$
$\top$	$\top$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$

defined in terms of conjunction and disjunction, as is possible in CPL. The complete algebraization of IPL, then, must somehow make use of this connective as an operator. The complete algebraization of IPL gives us what is known as a Heyting Algebra [5], which is a distributed lattice with an additional operator $\rightarrow$ with the following additional axioms that govern the behavior of the new operator:

1. $a \rightarrow a = \top$
2. $a \wedge (a \rightarrow b) = a \wedge b$
3. $b \wedge (a \rightarrow b) = b$
4. $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$

If we define the implication operation so that $a \rightarrow b = \neg a \wedge b$, then the Heyting algebra becomes a Boolean algebra. In this way, Boolean algebras are a subset of Heyting algebras; equivalently, we could say that CPL's are a subset of IPL's.

5 Closing Remarks

The goal of this paper was not to give a complete description of the relationship between logic systems and algebras but to give a hint at the deep connection between the two fields. The complete description of the correspondence between a class of algebras and their logical counterparts was given by Tarski in 1935 [2], who expanded on Lindenbaum's idea of viewing the equivalence classes of formulas as a set with operations given by the connectives in the language. The complete, generalized version of Tarski's method is known as an *algebraization* of a logic system. Today, the investigation of the process of algebraization itself, and inquiry into the classes of logics can be successfully algebraized, has given rise to the field of Abstract Algebraic Logic.

References

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