The Transfer Homomorphsim

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The transfer homomorphism from a group G to an abelian subgroup H of finite index in G arises from the natural action of G on the cosets of H. It plays an important role in the study of unsolvable groups and aids in the characterization of groups in which all Sylow p-subgroups are cyclic. In this paper, we will define the transfer homomorphism, and then investigate a few of its properties.

Let G be a group with an abelian subgroup H of finite index, $[G : H] = n$. G acts naturally on the right cosets of H (we will assume all of our cosets are right cosets). Consider a coset Hx and $g \in G$, where $Hx \mapsto Hxg$ under the action of G. Define a *transversal* for H in G to be a set of coset representatives. Let $\mathcal{T} = \{t_1, t_2 \dots t_n\}$ be a right transversal for H in G, with index set $\mathcal{J} = \{1, 2, \dots n\}$. Then G is equal to $\cup_{j \in \mathcal{J}} Ht_j$ and acts on $\mathcal{J}$.[1]

We are interested in a homomorphism defined by the action on the cosets when H is not normal in G. However, we cannot use the canonical homomorphism and mod by H. To solve this problem, observe that for $g \in G$, we can

[1] We are quite familiar with this action when H is normal in G. Supposing this case, let S be the subgroup of S_n containing permutations of $\mathcal{J}$ induced by the action of G. Then S is isomorphic to G/H. To see that this is so, let $\varphi : G/H \to S$ be defined by $Ht_i \mapsto s_i$, where s_i is the permutation of $\mathcal{J}$ induced by the action of $t_i \in G$. Clearly, φ is injective. To see that is surjective, consider $s \in S$. Then there exists $g \in G$ such that, under the action of right multiplication by g, $t_j^g = t_k$ whenever $s(j) = k$. Note that there is some $t_i \in \mathcal{T}$ such that $g \in Ht_i$. Observe, for $t_j \in \mathcal{T}$

$$Ht_j^g = Ht_jg$$

since $g \in Ht_i$, there is $h \in H$ such that

$$= Ht_jht_ig$$

and since $H \triangleleft G$, there exists $\hat{h} \in H$ such that

$$= H\hat{h}t_jt_i = Ht_jt_i = Ht_j^{t_i}.$$

Thus the action of s on $\mathcal{J}$ is induced by $t_i \in \mathcal{T}$, so $\varphi(t_i) = s$ and φ is surjective. To see that φ is an isomorphism, let $Ht_i, Ht_j \in G/H$ where $Ht_iHt_j = Ht_k$. Let $t_m \in T$. Then

$$\left(Ht_m^{t_i}\right)^{t_j} = Ht_mt_it_j = Ht_mt_k = Ht_m^{t^k}$$

so where s_i, the permutation of $\mathcal{J}$ induced by t_i, and s_j is the permutation induced by t_j, $s_is_j = s_k$. It is now straightforward to see

$$\varphi(Ht_iHt_j) = \varphi(Ht_it_j) = \varphi(Ht_k) = s_k = s_is_j = \varphi(Ht_i)\varphi(Ht_j)$$

and φ is an isomorphism.

define the action of G on $\mathcal{J}$, and our cosets, in the following natural way.

$$(Ht_j)^g = Ht_{j^g} = H(t_j g)$$

Moreover, under this action for all $j \in \mathcal{J}$ there exists a unique $h_{j,g} \in H$ such that

$$t_{j^g} = h_{j,g} t_j g.$$

Define the *transfer homomorphism* $\tau : G \to H$,

$$\tau(g) = \prod_{j \in \mathcal{J}} h_{j,g}.$$

Note that this product is well-defined as H is abelian. This somewhat unusual definition in fact gives rise to a homomorphism from G to H.

Proposition 1. *The transfer $\tau_{G \to H}$ is a homomorphism from G to H.*

Proof. Observe that for $a, b \in G$, $t_j \in \mathcal{T}$,

$$\begin{aligned} t_{j^{ab}} &= h_{j^a,b}(t_{j^a})b \\ &= h_{j^a,b}(h_{j,a} t_j a)b \\ &= h_{j^a,b} h_{j,a} t_j (ab). \end{aligned}$$

Then, we see

$$\begin{aligned} \tau(ab) &= \prod_{j \in \mathcal{J}} h_{j^a,b} h_{j,a} \\ &= \prod_{j \in \mathcal{J}} h_{j,a} \prod_{j \in \mathcal{J}} h_{j^a,b} \end{aligned}$$

so, recalling that a induces a permutation on $\mathcal{J}$,

$$\begin{aligned} &= \prod_{j \in \mathcal{J}} h_{j,a} \prod_{j \in \mathcal{J}} h_{j,b} \\ &= \tau(a)\tau(b). \end{aligned}$$

$\square$

While it may seem initially implausible that the transfer is indeed a homomorphism, there is another property which we require for τ to be well-defined.

Proposition 2. *The transfer $\tau_{G \to H}$ is independent of the choice of transversal.*

Proof. Let $\mathcal{S}$ and $\mathcal{T}$ be transversals for H in G, each indexed by $\mathcal{J}$. Recall our original definition $t_{j^g} = h_{j,g} t_j g$. We shall label this action $s_{j^g} = h^*_{j,g} s_j g$ for $h^* \in H$. Because $\mathcal{T}$ and $\mathcal{S}$ are transversals for the same subgroup of G, and therefore the same set of cosets, we can use the same index from $\mathcal{J}$ to index each. If $Ht_i = Hs_i$, then for $h \in H$, there exists $\hat{h} \in H$ such that $ht_i = \hat{h}s_i$. Therefore, $\hat{h}^{-1}ht_i = s_i$. Thus we have shown that for each $t_i \in \mathcal{T}$, there exists $\gamma_j \in H$ such that $\gamma_j t_j = s_j$. Recalling our initial equality, $t_{j^g} = h_{j,g}t_jg$, we proceed. Let $a \in G$.

$$\begin{aligned}\gamma_{j^a}(h_{j,a}t_ja) &= \gamma_{j^a}t_{j^a} \\ &= s_j^a \\ &= h^*_{j,a}s_ja \\ &= h^*_{j,a}\gamma_jt_ja\end{aligned}$$

This implies $h^*_{j,a} = \gamma_{j^a}h_{j,a}\gamma_j^{-1}$. Evaluating the transfer, τ^*, with respect to our transversal, S, we obtain

$$\begin{aligned}\tau^*(a) &= \prod_{j\in\mathcal{J}} h^*_{j,a} \\ &= \prod_{j\in\mathcal{J}} \gamma_{j^a}h_{j,a}\gamma_j^{-1}.\end{aligned}$$

Therefore, once again as a is a permutation of $\mathcal{J}$ and γ_i is in our abelian subgroup H,

$$\begin{aligned}&= \prod_{j\in\mathcal{J}} h_{j,a} \\ &= \tau(a).\end{aligned}$$

Which holds for all $a \in G$. Thus $\tau = \tau^*$, and the transfer homomorphism is independent of the transversal with which it is evaluated. □

In addition, the transfer homomorphism is transitive across nested subgroups.

Proposition 3. *Let G be a group, B an abelian subgroup of G with finite index and A a subgroup of B again with finite index (where $A \leq B$ is necessarily abelian). Then $\tau_{G\to A} = \tau_{B\to A} \circ \tau_{G\to B}$.*

Proof. Let $|G : B| = n$ and $\{x_1,\ x_2, \dots x_n\}$ be a transversal for B in G indexed by $\mathcal{J} = \{1, 2 \dots n\}$. Let $|B : A| = m$ and let $\{y_1,\ y_2, \dots y_m\} \subseteq B$ be a transversal for A in B indexed by $\mathcal{L} = \{1, 2 \dots m\}$. Then as $G = \cup_{j=1}^n Bx_i$ and $B = \cup_{l=1}^m Ay_l$,

$$G = \bigcup_{j\in\mathcal{J}} \left(\bigcup_{l\in\mathcal{L}} Ay_l \right) x_j.$$

That is $\{y_lx_j | l \in \mathcal{L}, j \in \mathcal{J}\}$ is a transversal for A in G if each coset Ay_lx_j is distinct. To see this is the case, suppose $Ayx = A\hat{y}\hat{x}$. Then $x\hat{x}^{-1} \in A\hat{y} \subseteq B$, and $Bx = B\hat{x}$. Since there is one representative per coset in our transversal,

$x = \hat{x}$, hence we see also $y = \hat{y}$. Thus each coset is distinct and we have a transversal for A in G. For $g \in G$, recall

$$\begin{aligned} x_{j^g} &= b_{jg} x_j g \\ &= a_{jlg} y_l x_j g. \end{aligned}$$

Note that for a particular a_{jlg} which is dependent upon j, l and g, there exists y_l such that $b_{jg} = a_{jgl} y_l$.

$$\tau_{G \to A}(g) = \prod_{j \in \mathcal{J}} \left(\prod_{l \in \mathcal{L}} a_{jlg} \right)$$

which, by the definition of the transfer $\tau_{B \to A}$

$$= \prod_{j \in \mathcal{J}} \tau_{B \to A}(b_{jg})$$

as the transfer is a homomorphism,

$$\begin{aligned} &= \tau_{B \to A} \left(\prod_{j \in \mathcal{J}} b_{jg} \right) \\ &= \tau_{B \to A} \circ \tau_{G \to B} \end{aligned}$$

again by the definition of the transfer. □

With some basic properties of the transfer in hand, we approach it on a computational level. With a clever choice of transversal, $\tau_{G \mapsto H}$ will not be as difficult to compute as it may first appear. Consider the action of G on its right cosets $\{Ht_1, Ht_2 \dots Ht_n\}$ by right multiplication. For $a \in G$, an orbit of a will have the form

$$\{Hs_i a, Hs_i a^2 \dots Hs_i a^{n_i - 1}\}$$

where n_i is the first power of a in Hs_i. If a has k distinct orbits, then $[G : H] = n = \sum_{i=1}^{k} n_i$. Note that $n_i | G$ as a consequence of the orbit-stabilizer theorem. Then $\{s_i a^j | 1 \leq i \leq k, 1 \leq j \leq n_j\}$ is a very useful transversal for computing the image of a. Observe that for $t \leq n_i - 1$,

$$(s_i a^t)^a = s_i a^{t+1} = h_{(s_i a^t, a)} s_i a^t a.$$

(Note we have slightly abused notation and indexed our subgroup element with a transversal element and a group element.) This implies that $h_{(s_i a^t, a)} = 1$. We now consider the case when $t = n_i - 1$.

$$s_i a^{n_i} = s_i = h_{(s_i a^{n_i - 1}, a)} s_i a^{n_i - 1} a$$

Therefore $h_{(s_i a^{n_i - 1}, a)} = s_i a^{n_i} s_i^{-1}$, which we see is in H as $H s_i a^{l_i} = H s_i$. Since in the i^{th} orbit it is only when $t = n_i - 1$ that $h_{(s_j a^t, a)} \neq 0$, we can evaluate our transfer

$$\begin{aligned}
\tau(a) &= \prod_{\substack{\{s_i a^j | 1 \leq i \leq k, \\ 1 \leq j \leq n_j\}}} h_{(s_i x^j, x)} \\
&= \prod_{i=1}^{k} \left(\prod_{j=0}^{n_i} h_{(s_i x^j, x)} \right) \\
&= \prod_{i=1}^{k} s_i a^{n_i} s_i^{-1}.
\end{aligned}$$

Thus, to compute $\tau(a)$, it suffices to find a representative for the i^{th} orbit of a's action upon the cosets of H, calling it s_i. When n_i is the size of the orbit, then take a^{n_i} and conjugate it by s_i. The product of these over all orbits gives $\tau(a)$. This has particularly interesting consequences when H is in the center of G.

Theorem 1. *Let $H \subseteq Z(G)$. Then $\tau_{G \to H}$ is an endomorphism of G given by the mapping $a \mapsto a^n$.*

Proof. Continuing with the notation from above, as $s_i a^{l_i} s_i^{-1} = h$ for some $h \in H$, it follows that

$$a^{l_i} = s_i^{-1} h s_i = s_i^{-1} s_i h = h$$

as $h \in Z(G)$. Thus $s_i a^{l_i} s_i^{-1} = a^{l_i}$. So

$$\begin{aligned}
\tau(a) &= \prod_{i=1}^{k} s_i a^{n_i} s_i^{-1} \\
&= \prod_{i=1}^{k} a^{n_i} \\
&= a^{\sum_{i=1}^{k} n_i} \\
&= a^n
\end{aligned}$$

As our selection of a was arbitrary, this is indeed our desired endomorphism. $\square$

The preceding result helps to develop an intuition why the transfer is employed in the abelianization of groups. Another such result follows. The elements of H in the image of the transfer have a curious relationship to elements of the normalizer of H in G.

Lemma 1. *Let $H \leq G$, H abelian and of finite index in G. Let $b \in N_G(H)$, and $a \in G$. Then $\tau(a)$ and b commute.*

Proof. Let $\mathcal{T} = \{t_1,\ t_2 \ldots t_n\}$, be a right transversal for H in G. For $b \in N_G(H)$, there exists $h^*_{ja} \in H$ such that $h_{ja} = bh^*_{ja}b^{-1}$. Let $x^*ja = h^*_{ja}x_ja$. Then $x^*_{ja} \in Hx_ja = Hx_{j^a}$, so for all j, $Hx_{ja} = Hx^*_{ja}$. That is, $\{x^*_{ja} | j \in \mathcal{J}\}$ is a transversal for H in G.

Moreover, as the image of the transfer is independent of our choice of transversal,

$$\tau(a) = \Pi_j h_{ja} = \Pi_j bh^*_{ja}b^{-1} = b\left(\Pi h^*_{ja}\right)b^{-1} = b\tau(a)b^{-1}$$

and thus $\tau(a)b = b\tau(a)$. $\square$

Ultimately, it is even possible to define the transfer without the assumption that H is itself abelian. In fact, if $\theta : H \to A$ is a homomorphism from H into some abelian group A, then we define the transfer from G into A

$$\tau_{G\to A}(a) = \prod_{i=1}^{k} \theta(s_i a^{n_i} s_i^{-1}).$$

This generalization has the advantage of creating an abelian quotient group. However, results such as the previous two, in which the image of τ is a subgroup of G fail. Either way, the transfer is a useful and unexpected tool of finite group theory. See [2] for further applications.

References

1. Péter Hermann, Advanced Abstract Algebra course notes, Budapest Semesters in Mathematics, Spring 2009.

2. Derek J. S. Robinson, "A Course in the Theory of Groups," Springer, New York, 1996.