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Exam 7 Spring 2010
Chapter R

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.

1. Consider the following linear transformation $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ and the bases B and C of $\mathbb{C}^2$. (30 points)

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a - b \\ -4a + 2b \end{bmatrix} \quad B = \left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}, \quad C = \left\{\begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$$

Solution:

(a) Compute $T\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right)$.

Solution:

$$T\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2(5) - 3 \\ -4(5) + 2(3) \end{bmatrix} = \begin{bmatrix} 7 \\ -14 \end{bmatrix}$$

(b) Construct the matrix representation of T relative to the bases B and C, $M^T_{B,C}$.

Solution:

$$\rho_B\left(T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)\right) = \rho_B\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \rho_B\left((-8)\begin{bmatrix} 4 \\ 3 \end{bmatrix} + (11)\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -8 \\ 11 \end{bmatrix}$$

$$\rho_B\left(T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\right) = \rho_B\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \rho_B\left((0)\begin{bmatrix} 4 \\ 3 \end{bmatrix} + (0)\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $M^T_{B,C} = \begin{bmatrix} -8 & 0 \\ 11 & 0 \end{bmatrix}$.

(c) Repeat the computation in the first part of this problem, carefully employing all of the relevant notation needed to illustrate the use of Theorem FTMR.

Solution:

$$\begin{aligned} T\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right) &= \rho_C^{-1}\left(M^T_{B,C}\rho_B\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right)\right) \\ &= \rho_C^{-1}\left(M^T_{B,C}\begin{bmatrix} 7 \\ -9 \end{bmatrix}\right) \\ &= \rho_C^{-1}\left(\begin{bmatrix} -56 \\ 77 \end{bmatrix}\right) \\ &= (-56)\begin{bmatrix} 4 \\ 3 \end{bmatrix} + (77)\begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ -9 \end{bmatrix} \end{aligned}$$

2. Consider the linear transformation $S\colon P_3 \to M_{22}$ defined below. (35 points)

$$S\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} a + 2b + c + 2d & -a - b - d \\ 2a + 3b + 2c + 5d & -a - c - 3d \end{bmatrix}$$

Solution:

(a) Compute a matrix representation of S.

Solution: With no bases specified, you might as well pick a couple of nice ones:

$$B = 1,\ x,\ x^2,\ x^3 \qquad\qquad C = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

A matrix representation relative to these bases should be straightforward at this stage:

$$M^S_{B,C} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & -1 & 0 & -1 \\ 2 & 3 & 2 & 5 \\ -1 & 0 & -1 & -3 \end{bmatrix}$$

(b) Use this matrix representation to determine that S is invertible by computing the kernel and range of the linear transformation. Be sure to explain the relevant theorems that allow you to begin with a matrix representation and finally arrive at the invertibility of the linear transformation. (No credit will be given for other methods.)

Solution: The matrix representation row-reduces as

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & -1 & 0 & -1 \\ 2 & 3 & 2 & 5 \\ -1 & 0 & -1 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Since the null space of the matrix representation is trivial (Theorem BNS), the kernel of the linear transformation is trivial (Theorem KNSI) and by Theorem KILT, S is injective. With a column space of dimension 4 (Theorem BCS), the range of the linear transformation has dimension 4 (Theorem RCSI) and is all of M_{22} (Theorem EDYES) and so S is surjective (Theorem RSLT). Finally, by Theorem ISLTI, we see S is an invertible linear transformation.

(c) Now that you have established that S is invertible, determine a formula for S^{-1}.

Solution: From Theorem IMR

$$M^{S^{-1}}_{C,B} = \left(M^S_{B,C}\right)^{-1} = \begin{bmatrix} 4 & -1 & -3 & -2 \\ -1 & 0 & 1 & 1 \\ 5 & 1 & -3 & -2 \\ -3 & 0 & 2 & 1 \end{bmatrix}$$

Applying Theorem FTMR

$$\begin{aligned}
S^{-1}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= \rho_B^{-1}\left(M^{S^{-1}}_{C,B}\,\rho_C\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right) \\
&= \rho_B^{-1}\left(M^{S^{-1}}_{C,B}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) \\
&= \rho_B^{-1}\left(\begin{bmatrix} 4a - b - 3c - 2d \\ -a + c + d \\ 5a + b - 3c - 2d \\ -3a + 2c + d \end{bmatrix}\right) \\
&= (4a - b - 3c - 2d) + (-a + c + d)x + (5a + b - 3c - 2d)x^2 + (-3a + 2c + d)x^3
\end{aligned}$$

3. Consider the linear transformation $R\colon P_2 \to P_2$ defined below. (35 points)

$R\left(a+bx+cx^2\right) = (-b-c) + (4a+4b+2c)x + (-2a-b+c)x^2$

Solution:

(a) Compute a matrix representation of R relative to a "nice" basis of P_2.

Solution: Notice that the basis you choose is being used for both the domain and codomain, say $F = \left\{1, x, x^2\right\}$. Then the matrix represenation should be straightforward:

$$M = M^R_{F,F} = \begin{bmatrix} 0 & -1 & -1 \\ 4 & 4 & 2 \\ -2 & -1 & 1 \end{bmatrix}$$

(b) Find a second basis of P_2 such that a matrix representation of R relative to this basis is a diagonal matrix and give the resulting representation.

Solution: Such a basis can be found from eigenvectors of (any) matrix representation, so we first compute eigenspaces.

$$\mathcal{E}_M(1) = \left\langle\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}\right\rangle \qquad \mathcal{E}_M(2) = \left\langle\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}\right\rangle$$

We now "un-coordinatize" each of these eigenvectors relative to the basis F to obtain the basis G below. We know this set is linearly independent since the algebraic and geometric multiplicities are equal for all the eigenspaces (since the geometric multiplicities sum to 3, the dimension of P_2), and since the size of the set is the dimension of P_2, by Theorem G th set is a basis.

$$G = \left\{1 - 2x + x^2,\ 1 - 2x^2,\ x - x^2\right\}$$

The matrix representation of R, relative to G is then a diagonal matrix with the eigenvalues on the diagonal (in the order of the eigenvectors in G),

$$M^R_{G,G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c) Compute a change-of-basis matrix between the two bases in the previous parts of this problem and show how to use it to convert between the two matrix representations, employing correct notation along the way.

Solution: It is easiest to express the basis of eigenvectors, G, in terms of the basis vectors in the "nice" basis F, this is the change-of-basis matrix $C_{G,F}$,

$$C_{G,F} = M^{I_{P_2}}_{G,F} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & -2 & -1 \end{bmatrix}$$

According to Theorem MRCB, and with an application of Theorem ICBM,

$$\begin{aligned} M^R_{G,G} &= C_{F,G} M^R_{F,F} C_{G,F} \\ &= C^{-1}_{G,F} M^R_{F,F} C_{G,F} \end{aligned}$$

which you could check computationally if you wished.