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Exam 5 Spring 2010
Chapter D&E

Show *all* of your work and *explain* your answers fully. There is a total of 95 possible points.

1. For the matrix A below, use row operations to create an upper-triangular matrix with 1's on the diagonal. Based on your sequence of row operations, compute the determinant of the matrix. No credit will be given for other methods. (15 points)

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 0 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 0 & 1 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 4 & -2 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & -4 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & -4 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{4}R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = B$$

Then

$$\det A = (-1)(1)(1)(4)(1)(3)\det(B) = -12(1) = -12$$

2. Find all the eigenvalues and all eigenspaces of the matrix B below, working all the necessary computations by hand. (15 points)

$$B = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$$

Solution: First the characteristic polynomial,

$$p_A(x) = \det B - xI_2 = \begin{vmatrix} -4-x & 6 \\ -3 & 5-x \end{vmatrix} = (-4-x)(5-x) - (-3)(6) = x^2 - x - 2 = (x-2)(x+1)$$

The roots of the characteristic polynomial are the eigenvalues (Theorem EMRCP), in this case $\lambda = 2, -1$. Now for eigenspaces,

$$\lambda = 2 \qquad B - 2I_2 = \begin{bmatrix} -6 & 6 \\ -3 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -1 \\ 0 & 0 \end{bmatrix} \qquad \mathcal{E}_B(2) = \mathcal{N}(B - 2I_2) = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = -1 \qquad F - (-1)I_2 = \begin{bmatrix} -3 & 6 \\ -3 & 6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -2 \\ 0 & 0 \end{bmatrix} \qquad \mathcal{E}_B(-1) = \mathcal{N}(B - (-1)I_2) = \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \right\rangle$$

3. The matrix F below has characteristic polynomial $p_A(x) = (x+6)(x-3)^3$. (20 points)

$$F = \begin{bmatrix} -16 & 20 & -21 & 41 \\ -27 & 30 & -27 & 54 \\ 15 & -12 & 12 & -21 \\ 16 & -14 & 12 & -23 \end{bmatrix}$$

Solution:

(a) Determine the algebraic and geometric multiplicities of the eigenvalues of F. Use your computing device only for row-reducing matrices.

Solution: From the factorization of the characteristic polynomial, we see that $\alpha_F(-6) = 1$ and $\alpha_F(3) = 3$. By Theorem ME, the geometric multiplicity of $\lambda = -6$ is trapped between 1 on the low side and $\alpha_F(-6) = 1$ on the high side, so without any computation, we see that $\gamma_F(-6) = 1$. We must compute however to determine the geometric multiplicity of $\lambda = -3$.

$$F - (-3)I_4 = \begin{bmatrix} -19 & 20 & -21 & 41 \\ -27 & 27 & -27 & 54 \\ 15 & -12 & 9 & -21 \\ 16 & -14 & 12 & -26 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenspace of $\lambda = -3$ is the null space of $F - (-3)I_4$. From the row-equivalent matrix in reduced row-echelon form, we see that a basis for this null space will have two vectors, owing to the two non-pivot columns. Thus $\gamma_F(3) = \dim(\mathcal{N}(F - (-3)I_4)) = 2$.

(b) Is F diagonalizable? Why or why not?

Solution: Since $\gamma_F(3) = 2 \neq 3 = \alpha_F(3)$, by Theorem DMFE, F is not diagonalizable.

4. Find a nonsingular matrix S and a diagonal matrix D so that $S^{-1}CS = D$. You may use your favorite computing device to get as much information about C's eigenvalues and eigenspaces as you like. (15 points)

$$C = \begin{bmatrix} 22 & -20 & 20 \\ 35 & -33 & 35 \\ 10 & -10 & 12 \end{bmatrix}$$

Solution: From our favorite computing device, we get the eigenvalues $\lambda = 2, 2, -3$, and some version of basis vectors for the eigenspaces,

$$\mathcal{E}_C(2) = \left\langle \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \right\rangle \qquad \mathcal{E}_C(-3) = \left\langle \left\{ \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix} \right\} \right\rangle$$

From the proof of Theorem DC, we construct a nonsingular matrix with columns from the basis vectors for the eigenspaces, and a diagonal matrix with the corresponding eigenvalues in the same order,

$$S = \begin{bmatrix} -1 & 1 & 4 \\ 0 & 1 & 7 \\ 1 & 0 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

and we *know* (i.e. we do not have to compute) that $C = S^{-1}CS$, though we could do a check.

5. Prove that an eigenspace is closed under scalar multiplication. In other words, if $\mathbf{x} \in \mathcal{E}_A(\lambda)$ and $\alpha \in \mathbb{C}$, then $\alpha\mathbf{x} \in \mathcal{E}_A(\lambda)$. (Provide a careful direct argument that does not rely on knowing that an eigenspace is a subspace.) (15 points)

   Solution: Suppose that $\alpha \in \mathbb{C}$, and that $\mathbf{x} \in \mathcal{E}_A(\lambda)$, that is, $\mathbf{x}$ is an eigenvector of A for λ. Then

$$\begin{aligned} A(\alpha\mathbf{x}) &= \alpha(A\mathbf{x}) && \text{Theorem MMSMM} \\ &= \alpha(\lambda\mathbf{x}) && \text{Definition EEM} \\ &= \lambda(\alpha\mathbf{x}) && \text{Property SMAC} \end{aligned}$$

   So by Definition EEM, $\alpha\mathbf{x}$ is an eigenvector of A for λ. Thus $\alpha\mathbf{x} \in \mathcal{E}_A(\lambda)$, and we have scalar closure.

6. Suppose that A is a square singular matrix, and B is any square matrix of the same size. Give a careful proof that AB is a singular matrix using certain basic properties of determinants. (No credit will be awarded for other proofs.) (15 points)

   Solution:

$$\begin{aligned} \det AB &= \det A \det B && \text{Theorem DRMM} \\ &= 0 \det B && \text{Theorem SMZD} \\ &= 0 && \text{Property ZCN} \end{aligned}$$

   Since AB has a zero determinant, it must be singular by Theorem SMZD.