Show all of your work and explain your answers fully. There is a total of 100 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. Solve the following system of equations using the inverse of a matrix. No credit will be given for solutions found with other methods. (15 points)

\[
\begin{align*}
  x_1 + 2x_2 + x_4 &= 0 \\
  x_1 + 3x_2 - x_3 + 3x_4 &= 6 \\
  2x_1 + 4x_2 + x_3 + x_4 &= -5 \\
  x_1 + 4x_2 + 4x_4 &= 6 \\
\end{align*}
\]

Solution: The coefficient matrix and vector of constants for this system are

\[
A = \begin{bmatrix}
  1 & 2 & 0 & 1 \\
  1 & 3 & -1 & 3 \\
  2 & 4 & 1 & 1 \\
  1 & 4 & 0 & 4
\end{bmatrix}, \quad b = \begin{bmatrix}
  0 \\
  6 \\
  -5 \\
  6
\end{bmatrix}
\]

So by Theorem SLEMM, the system can be re-expressed as \( Ax = b \), which by Theorem SNCM (presuming the coefficient matrix is nonsingular) has the unique solution

\[
x = A^{-1}b = \begin{bmatrix}
  12 & -4 & -4 & 1 \\
  -8 & 3 & 3 & -1 \\
  3 & -2 & -1 & 1 \\
  5 & -2 & -2 & 1
\end{bmatrix}
\begin{bmatrix}
  0 \\
  6 \\
  -5 \\
  6
\end{bmatrix} = \begin{bmatrix}
  2 \\
  -3 \\
  -1 \\
  4
\end{bmatrix}
\]

2. Determine if the vector \( u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is in the row space of the matrix \( B \) below. (15 points)

\[
B = \begin{bmatrix}
  2 & -1 & 3 \\
  1 & 4 & 2 \\
  3 & 3 & 5
\end{bmatrix}
\]

Solution: Notice that \( u \) is written as a column vector and the question is about membership in the row space.

\( u \in \mathcal{R}(B) \iff u \in \mathcal{C}(B^t) \iff \mathcal{L}(B^t, u) \) is consistent

So we consider the consistency of this system by row reducing the augmented matrix,

\[
\begin{bmatrix}
  2 & 1 & 3 & 1 \\
  -1 & 4 & 3 & 1 \\
  3 & 2 & 5 & 1
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
  1 & 0 & 1 & 0 \\
  0 & 1 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

By Theorem RCLS this system is inconsistent, so \( u \) is not in the row space of \( B \).
3. For the matrix $A$ below, in each part express the column space of $A$, $C(A)$, as the span of a linearly independent set satisfying the indicated conditions. (35 points)

$$A = \begin{bmatrix}
-3 & -1 & 1 & 4 & -1 \\
2 & 1 & -1 & -2 & 0 \\
-3 & 1 & 1 & 10 & -3 \\
-2 & 0 & 1 & 5 & -1
\end{bmatrix}$$

(a) Vectors in the spanning set are columns of $A$.

Solution: Theorem BCS is a rehash of Theorem BS: row-reduce the matrix, identify indices of pivot columns and use the columns of the original matrix with the same indices.

$$A \rightarrow \begin{bmatrix}
1 & 0 & 0 & -2 & 1 \\
0 & 1 & 0 & 3 & -1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

So $D = \{1, 2, 3\}$ and

$$S = \left\{ \begin{bmatrix}
-3 \\
2 \\
-3 \\
-2
\end{bmatrix}, \begin{bmatrix}
-1 \\
1 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix} \right\}$$

(b) Vectors in the spanning set begin with a “nice pattern of zeros and ones.”

Solution: The column space of $A$ is the row space of $A^t$ (Theorem CSRST). So transpose $A$, row reduce, and by Theorem BRS select the non-zero rows as columns vectors in $S$.

$$A^t = \begin{bmatrix}
-3 & 2 & -3 & -2 \\
-1 & 1 & 1 & 0 \\
1 & -1 & 1 & 1 \\
4 & -2 & 10 & 5 \\
-1 & 0 & -3 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & -\frac{1}{2} \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

So

$$S = \left\{ \begin{bmatrix}
1 \\
0 \\
0 \\
-\frac{1}{2}
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
0 \\
-1
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
1 \\
\frac{1}{2}
\end{bmatrix} \right\}$$

(c) Vectors in the spanning set end with a “nice pattern of zeros and ones.”

Solution: We form the extended echelon form of the matrix,

$$M = \begin{bmatrix}
-3 & -1 & 1 & 4 & -1 & 1 & 0 & 0 & 0 \\
2 & 1 & -1 & -2 & 0 & 0 & 1 & 0 & 0 \\
-3 & 1 & 1 & 10 & -3 & 0 & 0 & 1 & 0 \\
-2 & 0 & 1 & 5 & -1 & 0 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & -2 & 1 & 0 & 1 & -1 & 2 \\
0 & 1 & 0 & 3 & -1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 2 & -2 & 5 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & -1 & 2
\end{bmatrix}$$

The last row, in the last four columns, forms the matrix $L$, which is already in reduced row-echelon form

$$L = \begin{bmatrix}
1 & 2 & -1 & 2
\end{bmatrix}$$
and by Theorem FS, the null space equal of $L$ is equal to the column space of $A$, so we can apply Theorem BNS,
\[
\mathcal{C}(A) = \mathcal{N}(L) = \langle \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rangle
\]

4. Prove that a unitary matrix is nonsingular. This is part of the conclusion of Theorem UMI, so do more than just quote this result. (10 points)

Solution: If $A$ is a unitary matrix, then $A^*A = I_n$, or rephrased, $A^{-1} = A^*$, so in particular, $A$ is invertible (Definition MI, Theorem OSIS). An invertible matrix is nonsingular (Theorem NI).

Or, a unitary matrix has columns that form an orthonormal set (Theorem CUMOS). Every orthogonal set is linearly independent, so $A$ has linearly independent columns (Theorem OSLI). By Theorem NMLIC, $A$ is nonsingular.

5. For $m \times n$ matrices $A$ and $B$, prove that $A + B = B + A$. Include reasons for each step of your proof. (10 points)

Solution: This is Property CM. We work entry-by-entry, for $1 \leq i \leq m$, $1 \leq j \leq n$
\[
[A + B]_{ij} = [A]_{ij} + [B]_{ij} \quad \text{Definition MA}
\]
\[
= [B]_{ij} + [A]_{ij} \quad \text{Property CACN}
\]
\[
= [B + A]_{ij} \quad \text{Definition MA}
\]

So by Definition ME, the matrices $A + B$ and $B + A$ are equal.

6. Suppose that $A$ is an $m \times n$ matrix and $I_m$ is the size $m$ identity matrix. Write a careful proof that $I_mA = A$. (15 points)

Solution: This is Theorem MMIM. See a similar proof there.