Math 290 Name: Key Dr. Beezer
Quiz M Spring 2010

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. Solve the following system of equations using the inverse of a matrix. No credit will be given for solutions found with other methods. (15 points)

$$\begin{aligned} x_1 + 2x_2 \quad\quad + x_4 &= 0 \\ x_1 + 3x_2 - x_3 + 3x_4 &= 6 \\ 2x_1 + 4x_2 + x_3 + x_4 &= -5 \\ x_1 + 4x_2 \quad\quad + 4x_4 &= 6 \end{aligned}$$

Solution: The coefficient matrix and vector of constants for this system are

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 3 & -1 & 3 \\ 2 & 4 & 1 & 1 \\ 1 & 4 & 0 & 4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 6 \\ -5 \\ 6 \end{bmatrix}$$

So by Theorem SLEMM, the system can be re-expressed as $A\mathbf{x} = \mathbf{b}$, which by Theorem SNCM (presuming the coefficient matrix is nonsingular) has the unique solution

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 12 & -4 & -4 & 1 \\ -8 & 3 & 3 & -1 \\ 3 & -2 & -1 & 1 \\ 5 & -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \\ 4 \end{bmatrix}$$

2. Determine if the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the row space of the matrix B below. (15 points)

$$B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 2 \\ 3 & 3 & 5 \end{bmatrix}$$

Solution: Notice that $\mathbf{u}$ is written as a column vector and the question is about membership in the row space.

$$\mathbf{u} \in \mathcal{R}(B) \iff \mathbf{u} \in \mathcal{C}\left(B^t\right) \iff \mathcal{LS}\left(B^t,\, \mathbf{u}\right) \text{ is consistent}$$

So we consider the consistency of this system by row reducing the augmented matrix,

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

By Theorem RCLS this system is inconsistent, so $\mathbf{u}$ is not in the row space of B.

3. For the matrix A below, in each part express the column space of A, $\mathcal{C}(A)$, as the span of a linearly independent set satisfying the indicated conditions. (35 points)

$$A = \begin{bmatrix} -3 & -1 & 1 & 4 & -1 \\ 2 & 1 & -1 & -2 & 0 \\ -3 & 1 & 1 & 10 & -3 \\ -2 & 0 & 1 & 5 & -1 \end{bmatrix}$$

(a) Vectors in the spanning set are columns of A.

Solution: Theorem BCS is a rehash of Theorem BS: row-reduce the matrix, identify indices of pivot columns and use the columns of the original matrix with the same indices.

$$\begin{bmatrix} -3 & -1 & 1 & 4 & -1 \\ 2 & 1 & -1 & -2 & 0 \\ -3 & 1 & 1 & 10 & -3 \\ -2 & 0 & 1 & 5 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -2 & 1 \\ 0 & \boxed{1} & 0 & 3 & -1 \\ 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $D = \{1, 2, 3\}$ and

$$S = \left\{ \begin{bmatrix} -3 \\ 2 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(b) Vectors in the spanning set begin with a "nice pattern of zeros and ones."

Solution: The column space of A is the row space of A^t (Theorem CSRST). So transpose A, row reduce, and by Theorem BRS select the non-zero rows as columns vectors in S.

$$A^t = \begin{bmatrix} -3 & 2 & -3 & -2 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 1 \\ 4 & -2 & 10 & 5 \\ -1 & 0 & -3 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -\frac{1}{2} \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{1}{2} \end{bmatrix} \right\}$$

(c) Vectors in the spanning set end with a "nice pattern of zeros and ones."

Solution:

We form the extended echelon form of the matrix,

$$M = \begin{bmatrix} -3 & -1 & 1 & 4 & -1 & 1 & 0 & 0 & 0 \\ 2 & 1 & -1 & -2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 1 & 1 & 10 & -3 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 5 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -2 & 1 & 0 & 1 & -1 & 2 \\ 0 & \boxed{1} & 0 & 3 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 & 1 & 0 & 2 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 2 & -1 & 2 \end{bmatrix}$$

The last row, in the last four columns, forms the matrix L, which is already in reduced row-echelon form

$$L = \begin{bmatrix} \boxed{1} & 2 & -1 & 2 \end{bmatrix}$$

and by Theorem FS, the null space equal of L is equal to the column space of A, so we can apply Theorem BNS,

$$\mathcal{C}(A) = \mathcal{N}(L) = \left\langle \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

4. Prove that a unitary matrix is nonsingular. This is part of the conclusion of Theorem UMI, so do more than just quote this result. (10 points)

   Solution: If A is a unitary matrix, then $A^*A = I_n$, or rephrased, $A^{-1} = A^*$, so in particular, A is invertible (Definition MI, Theorem OSIS). An invertible matrix is nonsingular (Theorem NI).

   Or, a unitary matrix has columns that form an orthonormal set (Theorem CUMOS). Every orthogonal set is linearly independent, so A has linearly independent columns (Theorem OSLI). By Theorem NMLIC, A is nonsingular.

5. For $m \times n$ matrices A and B, prove that $A + B = B + A$. Include reasons for each step of your proof. (10 points)

   Solution: This is Property CM. We work entry-by-entry, for $1 \leq i \leq m$, $1 \leq j \leq n$

$$\begin{aligned} [A+B]_{ij} &= [A]_{ij} + [B]_{ij} && \text{Definition MA} \\ &= [B]_{ij} + [A]_{ij} && \text{Property CACN} \\ &= [B+A]_{ij} && \text{Definition MA} \end{aligned}$$

   So by Definition ME, the matrices $A + B$ and $B + A$ are equal.

6. Suppose that A is an $m \times n$ matrix and I_m is the size m identity matrix. Write a careful proof that $I_mA = A$. (15 points)

   Solution: This is Theorem MMIM. See a similar proof there.