Show all of your work and explain your answers fully. There is a total of 100 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. For the system below, compute the inverse of the nonsingular coefficient matrix by row-reducing the appropriate $2 \times 4$ matrix. Then use this inverse to compute the solution set of the system. (10 points)

$$
3x_1 + 2x_2 = 5 \\
11x_1 + 7x_2 = 19
$$

Solution: Augment the coefficient matrix of the system with the $2 \times 2$ identity matrix. By the combination of Theorem CINM and Theorem OSIS, we can find the inverse from the row-reduced version of this matrix,

$$
\begin{bmatrix}
3 & 2 & 1 & 0 \\
11 & 7 & 0 & 1
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & -7 & 2 \\
0 & 1 & 11 & -3
\end{bmatrix}
$$

If we use Theorem SLEMM to rewrite the system as $Ax = b$, then Theorem SNCM tells us the solution set contains a single vector,

$$
x = A^{-1}b = \begin{bmatrix}
-7 \\
11
\end{bmatrix}
\begin{bmatrix}
5 \\
19
\end{bmatrix} = \begin{bmatrix}
3 \\
-2
\end{bmatrix}
$$

2. For the matrix $A$ below, demonstrate the use of Theorem FS to compute the four indicated sets. (30 points)

$$
A = \begin{bmatrix}
-5 & 13 \\
2 & -5 \\
-1 & 3
\end{bmatrix}
$$

(a) Null space of $A$, $N(A)$.

Solution: For all four parts of this problem, we need the extended echelon form of $A$ (Definition EEF), and we extract the submatrices $C$ and $L$.

$$
\begin{bmatrix}
-5 & 13 & 1 & 0 & 0 \\
2 & -5 & 0 & 1 & 0 \\
-1 & 3 & 0 & 0 & 1
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 & 3 & 5 \\
0 & 1 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 & -1
\end{bmatrix}
$$

$C = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}$, $L = \begin{bmatrix}
1 & 2 & -1
\end{bmatrix}$

Using Theorem FS and Theorem BNS, $N(A) = N(C) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

(b) Row space of $A$, $\mathcal{R}(A)$.

Solution: Using Theorem FS and Theorem BRS, $\mathcal{R}(A) = \mathcal{R}(C) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \right\rangle = \mathbb{C}^2$.

(c) Column space of $A$, $\mathcal{C}(A)$.

Solution: Using Theorem FS and Theorem BNS, $\mathcal{C}(A) = \mathcal{N}(L) = \left\langle \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$. 
(d) Left null space of $A$, $\mathcal{L}(A)$.

Solution: Using Theorem FS and Theorem BRS, $\mathcal{L}(A) = \mathcal{R}(L) = \left\{ \left[ \begin{array}{c} 1 \\ 2 \\ -1 \end{array} \right] \right\}$.

For Problems 3 and 4 use the following vectors and matrix. 15 points for each problem.

$$w = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad y = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ -2 \\ 1 \\ 0 \\ 2 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -1 & 1 & 0 & -3 \\ -3 & 2 & 0 & -1 & 4 \\ 4 & -3 & -1 & 2 & -5 \\ -5 & 4 & 2 & -3 & 6 \end{bmatrix}$$

3. (a) Prove directly (without using other parts of this problem) that $w$ is an element of the column space of $A$, $w \in \mathcal{C}(A)$.

Solution: By Theorem CSCS, $w \in \mathcal{C}(A)$ if and only if $\mathcal{L}(A, w)$ is consistent. We row-reduce the augmented matrix of this system

$$[A | w] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & -1 & -2 & 3 \\ 0 & 1 & 3 & -2 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem RCLS this system is consistent, so $w \in \mathcal{C}(A)$.

(b) Find a linearly independent set $S$ such that the span of $S$ is the column space of $A$, $\mathcal{C}(A) = \langle S \rangle$.

Solution: We can see the reduced row-echelon form of $A$ in the first 5 columns of the row-reduced matrix in the previous part of this problem. In particular, $D = \{1, 2\}$. By Theorem BCS we can take columns 1 and 2 of $A$ as column vectors and this set will fulfill the requirements of the question.

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ -1 \\ 4 \end{bmatrix} \right\}$$

(c) Write $w$ as a linear combination of the elements of $S$.

Solution: We can use the row-reduced matrix in part (a) to construct solutions to the linear system $\mathcal{L}(A, w)$. By Theorem SLSLC these solutions will give rise to linear combinations of the columns of $A$ that equal $x$. In this case we want a linear combination of just the first two columns of $A$, so we want the scalars for the last three columns to be zero, i.e. $x_3 = x_4 = x_5 = 0$. Not coincidentally, the last three variables in the system are free, so we can just choose them to be zero. The result is that the first two variables are $x_1 = 3$ and $x_2 = 4$, so we have

$$3 \begin{bmatrix} 2 \\ -3 \\ 4 \\ -5 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$
4. (a) Prove directly (without using other parts of this problem) that $y$ is an element of the row space of $A$, $y \in \mathcal{R}(A)$.

Solution: By Definition RSM, $y \in \mathcal{R}(A)$ if and only if $y \in \mathcal{C}(A^t)$. We can test this by examining the consistency of $\mathcal{L}S(A^t, y)$ (Theorem CSCS). We row-reduce the augmented matrix,

$$\begin{bmatrix} 1 & 0 & -1 & 2 & -2 \\ 0 & 1 & -2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem RCLS this system is consistent, so $y \in \mathcal{R}(A)$.

(b) Find a linearly independent set $T$ such that the span of $T$ is the row space of $A$, $\mathcal{R}(A) = \langle T \rangle$.

Solution: Theorem BRS tells us we can row-reduce $A$, and take the nonzero rows as column vectors to form $T$ with the requested properties.

$$\begin{bmatrix} 1 & 0 & 2 & -1 & -2 \\ 0 & 1 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \\ -1 \end{bmatrix} \right\}$$

(c) Write $y$ as a linear combination of the elements of $T$.

Solution: The first two entries of each vector in $T$ will suggest the scalars $a_1 = -1$, $a_1 = 0$,

$$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ -3 \\ -1 \end{bmatrix}$$

5. Suppose that $A$ is a $m \times n$ matrix and $\alpha \in \mathbb{C}$ is a scalar. Prove that $(\alpha A)^t = \alpha A^t$ (Note: this is Theorem TMSM, so you are being asked to do more than just quote this result from the book.) (15 points)

Solution: See the proof of Theorem TMSM in the book.

6. Prove the converse of Theorem NPNT: If $A$ and $B$ are nonsingular matrices of size $n$, then $AB$ is a nonsingular matrix of size $n$. (15 points)

Solution: Suppose that $x \in \mathbb{C}^n$ is a solution to $\mathcal{L}S(AB, 0)$. Then

$$0 = (AB)x \quad \text{Theorem SLEMM}$$

$$= A(Bx) \quad \text{Theorem MMA}$$

By Theorem SLEMM, $Bx$ is a solution to $\mathcal{L}S(A, 0)$, and by the definition of a nonsingular matrix (Definition NM), we conclude that $Bx = 0$. Now, by an entirely similar argument, the nonsingularity of $B$ forces us to conclude that $x = 0$. So the only solution to $\mathcal{L}S(AB, 0)$ is the zero vector and we conclude that $AB$ is nonsingular.