Math 290 Name: Key Dr. Beezer
Quiz SLE Code: Spring 2008

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator or software package on a problem be sure to write down both the input and output.

1. Find all solutions to the system of equations below. Express the solutions as a set of column vectors. (15 points)

$$\begin{aligned} 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 + 2x_2 + x_3 &= 9 \\ 4x_1 - 3x_2 + 2x_3 &= 7 \end{aligned}$$

Solution: We form the augmented matrix of the system and row-reduce,

$$\begin{bmatrix} 2 & -1 & 4 & 7 \\ 3 & 2 & 1 & 9 \\ 4 & -3 & 2 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

With no leading 1 in the final column, this system is consistent (Theorem RCLS). There are $n = 3$ variables in the system and $r = 3$ non-zero rows in the row-reduced matrix. By Theorem FVCS, there are $n-r = 3-3 = 0$ free variables and we therefore know the solution is unique. Forming the system of equations represented by the row-reduced matrix, we see that $x_1 = 2$, $x_2 = 1$ and $x_3 = 1$. Written as a set of column vectors,

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

2. Find all solutions to the system of equations below. Express the solutions as a set of column vectors. (15 points)

$$\begin{aligned} -2x_1 + 2x_2 + x_3 - 3x_4 &= -10 \\ x_1 - x_2 \qquad + 2x_4 &= 3 \\ 3x_1 - 3x_2 + 4x_3 + 10x_4 &= 5 \end{aligned}$$

Solution: We form the augmented matrix of the system and row-reduce

$$\begin{bmatrix} -2 & 2 & 1 & -3 & -10 \\ 1 & -1 & 0 & 2 & 3 \\ 3 & -3 & 4 & 10 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -1 & 0 & 2 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

With a leading 1 in the final column, this system is inconsistent (Theorem RCLS). Written as a set, $S = \{\,\}$.

3. Find a matrix that is row-equivalent to C and in reduced row-echelon form. Perform the necessary computations by hand, not with a calculator, and show your intermediate steps along with the row operations you've performed. (15 points)

$$C = \begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & 4 \\ -1 & -2 & 3 & 5 \end{bmatrix}$$

Solution: Following the algorithm of Theorem REMEF, and working to create pivot columns from left to right, we have

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & 4 \\ -1 & -2 & 3 & 5 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 6 \\ -1 & -2 & 3 & 5 \end{bmatrix} \xrightarrow{1R_1+R_3} \begin{bmatrix} \boxed{1} & 2 & -1 & -1 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{1R_2+R_1}$$

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} \boxed{1} & 2 & 0 & 5 \\ 0 & 0 & \boxed{1} & 6 \\ 0 & 0 & 0 & -8 \end{bmatrix} \xrightarrow{-\frac{1}{8}R_3} \begin{bmatrix} \boxed{1} & 2 & 0 & 5 \\ 0 & 0 & \boxed{1} & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-6R_3+R_2}$$

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 5 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-5R_3+R_1} \begin{bmatrix} \boxed{1} & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

4. Find the null space of the matrix E below. (15 points)

$$E = \begin{bmatrix} 2 & 1 & -1 & -9 \\ 2 & 2 & -6 & -6 \\ 1 & 2 & -8 & 0 \\ -1 & 2 & -12 & 12 \end{bmatrix}$$

Solution: We form the augmented matrix of the homogeneous system $\mathcal{LS}(E,\,\mathbf{0})$ and row-reduce the matrix,

$$\begin{bmatrix} 2 & 1 & -1 & -9 & 0 \\ 2 & 2 & -6 & -6 & 0 \\ 1 & 2 & -8 & 0 & 0 \\ -1 & 2 & -12 & 12 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & -6 & 0 \\ 0 & \boxed{1} & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We knew ahead of time that this system would be consistent (Theorem HSC), but we can now see there are $n - r = 4 - 2 = 2$ free variables, namely x_3 and x_4 since $F = \{3, 4, 5\}$ (Theorem FVCS). Based on this analysis, we can rearrange the equations associated with each nonzero row of the reduced row-echelon form into an expression for the lone dependent variable as a function of the free variables. We arrive at the solution set to the homogeneous system, which is the null space of the matrix by Definition NSM,

$$\mathcal{N}(B) = \left\{ \begin{bmatrix} -2x_3 + 6x_4 \\ 5x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle|\, x_3,\, x_4 \in \mathbb{C} \right\}$$

5. Find all six-digit numbers in which the first digit is one less than the second, the third digit is half the second, the fourth digit is three times the third and the last two digits are the sum of the fourth and fifth. The sum of all the digits is 24. Use a system of equations to aid you in your search for this answer, no credit will be given for solutions attained by other methods. (From *The MENSA Puzzle Calendar* for January 9, 2006.) (15 points)

   Solution: Let $abcdef$ denote the six-digit number and convert each requirement in the problem statement into an equation.

$$\begin{aligned} a &= b - 1 \\ c &= \frac{1}{2}b \\ d &= 3c \\ e + f &= d + e \\ 24 &= a + b + c + d + e + f \end{aligned}$$

   In a more standard form this becomes

$$\begin{aligned} a - b &= -1 \\ -b + 2c &= 0 \\ -3c + d &= 0 \\ -d + f &= 0 \\ a + b + c + d + e + f &= 24 \end{aligned}$$

   Using row operations on an augmented matrix, this system can be converted to the equivalent system

$$\begin{aligned} a - \frac{2}{3}f &= -1 \\ b - \frac{2}{3}f &= 0 \\ c - \frac{1}{3}f &= 0 \\ d - f &= 0 \\ e + \frac{11}{3}f &= 25 \end{aligned}$$

   Clearly, we must choose the free variable f to be a multiple of 3, and of the choices $f = 0,\ 3,\ 6,\ 9$ only $f = 6$ results in a sensible (positive, single-digit) value for e. So with $f = 6$ we have

$$e = 3 \qquad d = 6 \qquad c = 2 \qquad b = 4 \qquad a = 3$$

   So the only such number is 342636. Notice that the question casts the numbers as digits, but their role as place values is not relevant.

6. Suppose that A is a nonsingular matrix and A is row-equivalent to the matrix B. Prove that B is nonsingular. (15 points)

   Solution: Since A and B are row-equivalent matrices, consideration of the three row operations (Definition RO) will show that the augmented matrices, $[A \,|\, \mathbf{0}]$ and $[B \,|\, \mathbf{0}]$, are also row-equivalent matrices. This says that the two homogeneous systems, $\mathcal{LS}(A,\, \mathbf{0})$ and $\mathcal{LS}(B,\, \mathbf{0})$ are equivalent systems. $\mathcal{LS}(A,\, \mathbf{0})$ has only the zero vector as a solution (Definition NM), thus $\mathcal{LS}(B,\, \mathbf{0})$ has only the zero vector as a solution. Finally, by Definition NM, we see that B is nonsingular.