Chapter 1

If $a$ and $b$ are integers, then we know there exist integers $r$ and $s$ so that $ra + sb = \gcd(a, b)$. Prove that the numbers $r$ and $s$ are not unique by showing that there are infinitely many pairs of integers, $(r, s)$, such that $ra + sb = \gcd(a, b)$.

Chapter 2

Compute the centers of some small (nonabelian) groups.

For certain elements of a (nonabelian) group, compute the centralizer.

For certain subgroups of a (nonabelian) group, compute the normalizer.

When is the centralizer of an element the trivial subgroup? When is it not the trivial subgroup?

Suppose $G$ is a group and $g \in G$. Show that $\langle g \rangle = \{g^m \mid m \in \mathbb{Z}\}$ is a subgroup of the centralizer $C(g)$.

Suppose that $H$ is a subgroup of $G$. Choose $g \in G$ and define $K_g = \{ghg^{-1} \mid h \in H\}$. Prove that $K_g$ is a subgroup of $G$. Describe $K_g$ when $g \in H$. Describe $K_g$ when $H$ is abelian. Describe $K_g$ when $G$ is abelian.

Chapter 5

The set of left cosets of the subgroup $H$ in the group $G$ forms a partition. Therefore, there is an associated equivalence relation defined on the set $G$. Describe this equivalence relation without using cosets in your final definition.

Chapter 8

Find a counterexample to the following assertion.

If $K$ is a subgroup of $G_1 \times G_2$, then $K = H_1 \times H_2$ where $H_1$ is a subgroup of $G_1$ and $H_2$ is a subgroup of $G_2$. (So the converse of problem 52 is false.)

Group of Units Revealed

Definition When $s|n$, define $U_s(n) = \{m \in U(n) \mid m \pmod{s} = 1\}$.

Fact If $m$ and $n$ are relatively prime, then $U_n(mn)$ is a subgroup of $U(mn)$, and $U_n(mn) \simeq U(m)$.

Fact If $m$ and $n$ are relatively prime, then $U(mn)$ is isomorphic to $U(m) \times U(n)$. 

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Proof  Define $\phi : U(mn) \to U(m) \times U(n)$ by

$$\phi(x) = (x \mod m, x \mod n)$$

Then show $\phi$ is an isomorphism.

Fact  $U(2) \simeq \{0\}$, $U(4) \simeq \mathbb{Z}_2$, $U(2^m) \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$. For a prime $p > 2$, $U(p^m) \simeq \mathbb{Z}_{p^m-p^{m-1}}$.

Example  $U(36) = U(2^23^2) \simeq U(4) \times U(9) \simeq \mathbb{Z}_2 \times \mathbb{Z}_6$

As an internal direct product, use subgroups $U_9(36)$ and $U_4(36)$.

Problems  Describe $U(72)$, $U(105)$ and $U(1350)$.

Chapter 9

Without doing the necessary computations, argue that $\{(), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$ is a normal subgroup of $S_4$. (Recall Problem 4.30)

Suppose that $G$ is a group and $H$ and $K$ are normal subgroups of $G$, such that

1. $G = HK = \{hk \mid h \in H, k \in K\}$
2. $H \cap K = \{e\}$

Prove that $G$ is an internal direct product of $H$ and $K$.

Chapter 11

$H = \{1, 22, 27, 29, 36, 43, 48, 55, 62, 64, 69, 90\}$ is a subgroup of $U(91)$. Determine a group that is isomorphic to $H$ and that is written as an external direct product of cyclic groups.

The group $G = U(63 \cdot 767)$ has order $\phi(63 \cdot 767) = 52,800$. In this problem you will build a subgroup of order 80 isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{10}$.

(a)  $52800 = 2^6 \cdot 3 \cdot 5^2 \cdot 11$. So we know $G$ can be written as a product of subgroups of orders $2^6$, $3$, $5^2$, $11$, say $H_{64}$, $H_3$, $H_{25}$, $H_{11}$. Compute each of these subgroups. For example, $H_{25} = \{x \in G \mid x^{25} = 1\}$. As a check, $H_3 = \{1, 10286, 12343\}$.

(b)  From information above, we can determine the eventual structure of $G$. We have $63 \cdot 767 = 11^2 \cdot 17 \cdot 31$, so

$$G \simeq \mathbb{Z}_{(11^2-11)} \times \mathbb{Z}_{16} \times \mathbb{Z}_{30}$$

$$\simeq \mathbb{Z}_{110} \times \mathbb{Z}_{16} \times \mathbb{Z}_{30}$$

$$\simeq \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_{11} \times \mathbb{Z}_{16} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$$

$$\simeq \mathbb{Z}_{16} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{11}$$
So in particular, we know in advance the structure of each of the $p$-groups:

\begin{align*}
H_{64} &\cong \mathbb{Z}_{16} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \\
H_3 &\cong \mathbb{Z}_3 \\
H_{25} &\cong \mathbb{Z}_5 \times \mathbb{Z}_5 \\
H_{11} &\cong \mathbb{Z}_{11}
\end{align*}

From this, explain how you know $G$ has a subgroup of order 80 isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \cong \mathbb{Z}_8 \times \mathbb{Z}_{10}$.

(c) Construct the group described in part (b).