Chapter 1

If a and b are integers, then we know there exist integers r and s so that $ra + sb = \gcd(a, b)$. Prove that the numbers r and s are not unique by showing that there are infinitely many pairs of integers, (r, s), such that $ra + sb = \gcd(a, b)$.

Chapter 2

Compute the centers of some small (nonabelian) groups.

For certain elements of a (nonabelian) group, compute the centralizer.

For certain subgroups of a (nonabelian) group, compute the normalizer.

When is the centralizer of an element the trivial subgroup? When is it not the trivial subgroup?

Suppose G is a group and $g \in G$. Show that $\langle g \rangle = \{g^m \mid m \in \mathbb{Z}\}$ is a subgroup of the centralizer $C(g)$.

Suppose that H is a subgroup of G. Choose $g \in G$ and define $K_g = \{ghg^{-1} \mid h \in H\}$. Prove that K_g is a subgroup of G. Describe K_g when $g \in H$. Describe K_g when H is abelian. Describe K_g when G is abelian.

Chapter 5

The set of left cosets of the subgroup H in the group G forms a partition. Therefore, there is an associated equivalence relation defined on the set G. Describe this equivalence relation without using cosets in your final definition.

Chapter 8

Find a counterexample to the following assertion.
If K is a subgroup of $G_1 \times G_2$, then $K = H_1 \times H_2$ where H_1 is a subgroup of G_1 and H_2 is a subgroup of G_2. (So the converse of problem 52 is false.)

Group of Units Revealed

Definition When $s|n$, define $U_s(n) = \{m \in U(n) \mid m \pmod{s} = 1\}$.

Fact If m and n are relatively prime, then $U_n(mn)$ is a subgroup of $U(mn)$, and $U_n(mn) \simeq U(m)$.

Fact If m and n are relatively prime, then $U(mn)$ is isomorphic to $U(m) \times U(n)$.

Proof Define $\phi : U(mn) \to U(m) \times U(n)$ by

$$\phi(x) = (x \mod m, x \mod n)$$

Then show ϕ is an isomorphism.

Fact $U(2) \simeq \{0\}$, $U(4) \simeq \mathbb{Z}_2$, $U(2^m) \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$. For a prime $p > 2$, $U(p^m) \simeq \mathbb{Z}_{p^m - p^{m-1}}$.

Example $U(36) = U(2^2 3^2) \simeq U(4) \times U(9) \simeq \mathbb{Z}_2 \times \mathbb{Z}_6$
As an internal direct product, use subgroups $U_9(36)$ and $U_4(36)$.

Problems Describe $U(72)$, $U(105)$ and $U(1350)$.

Chapter 9

Without doing the necessary computations, argue that $\{(), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ is a normal subgroup of S_4. (Recall Problem 4.30)

Suppose that G is a group and H and K are *normal* subgroups of G, such that

1. $G = HK = \{hk \mid h \in H, k \in K\}$
2. $H \cap K = \{e\}$

Prove that G is an internal direct product of H and K.

Chapter 11

$H = \{1, 22, 27, 29, 36, 43, 48, 55, 62, 64, 69, 90\}$ is a subgroup of $U(91)$. Determine a group that is isomorphic to H and that is written as an external direct product of cyclic groups.

The group $G = U(63\,767)$ has order $\phi(63\,767) = 52\,800$. In this problem you will build a subgroup of order 80 isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{10}$.

(a) $52800 = 2^6 \cdot 3 \cdot 5^2 \cdot 11$. So we know G can be written as a product of subgroups of orders 2^6, 3, 5^2, 11, say H_{64}, H_3, H_{25}, H_{11}. Compute each of these subgroups. For example, $H_{25} = \{x \in G \mid x^{25} = 1\}$. As a check, $H_3 = \{1,\ 10286,\ 12343\}$.

(b) From information above, we can determine the eventual structure of G. We have $63\,767 = 11^2 \cdot 17 \cdot 31$, so

$$\begin{aligned} G &\simeq \mathbb{Z}_{(11^2-11)} \times \mathbb{Z}_{16} \times \mathbb{Z}_{30} \\ &\simeq \mathbb{Z}_{110} \times \mathbb{Z}_{16} \times \mathbb{Z}_{30} \\ &\simeq \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_{11} \times \mathbb{Z}_{16} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\ &\simeq \mathbb{Z}_{16} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{11} \end{aligned}$$

So in particular, we know in advance the structure of each of the p-groups:

$$H_{64} \simeq \mathbb{Z}_{16} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad H_3 \simeq \mathbb{Z}_3 \qquad H_{25} \simeq \mathbb{Z}_5 \times \mathbb{Z}_5 \qquad H_{11} \simeq \mathbb{Z}_{11}$$

From this, explain how you know G has a subgroup of order 80 isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \simeq \mathbb{Z}_8 \times \mathbb{Z}_{10}$.

(c) Construct the group described in part (b).