Gershgorin's Theorem for Estimating Eigenvalues

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One of the most important things you can know about a matrix is its eigenvalue (or characteristic value). By pure inspection it is nearly impossible to see the eigenvalues. One way for estimating eigenvalues is to find the trace of the matrix. The trace merely tells us what all the eigenvalues add up to. It doesn't give us any range for the eigenvalues. Even if we have a very small trace we can still theoretically have two eigenvalues whose absolute values are very large but have an opposite sign. In order to figure out what range the eigenvalues of a certain matrix would be in we can use Gershgorin's Theorem.

1 Strictly Diagonally Dominant Matrices

Before we get to Gershgorin's Theorem it is convenient to introduce a condition for matrices known as Strictly Diagonally Dominant. While Gershgorin's Theorem can be proven by other means, it is simplest to solve it using knowledge of Strictly Diagonally Dominant matrices.

A Strictly Diagonally Dominant, here on referred to as SDD, matrix is defined as follows:

Definition 1 (Strictly Diagonally Dominant Matrices)

Matrix A_{nn} is Strictly Diagonally Dominant if: $|A_{ii}| > \sum_{j \neq i} |A_{ij}|$ for i = 1,2,...,n

Example 1

$$A= \begin{bmatrix} 6 & -1 & 2 & 2 \\ -1 & 5 & -1 & 2 \\ 1 & 1 & -8 & 5 \\ -1 & 0 & 0 & 3 \end{bmatrix}$$

It is fairly easy to see that each row satisfies the inequality $|A_{ii}| > \sum_{j \neq i} |A_{ij}|$

$$\begin{array}{lccccccc} \text{row 1} & |6| & > & |-1| & + & |2| & + & |2| \\ \text{row 2} & |5| & > & |-1| & + & |-1| & + & |2| \\ \text{row 3} & |-8| & > & |1| & + & |1| & + & |5| \\ \text{row 4} & |3| & > & |-1| & + & |0| & + & |0 \end{array}$$

Theorem 1.1 *(Nonsingularity of SDD Matrices)*

Strictly diagonally dominant matrices are always nonsingular.

Proof

□

Suppose that matrix A_{nn} is SDD and singular, then there exists a $u \in u_n$ such that $Au = b$ where b is the 0 vector while $u \neq 0$ (Definition NM[67]).

$$\text{A} = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ ... & & & ... \\ A_{n1} & A_{n2} & ... & A_{nn} \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ u_2 \\ ... \\ u_n \end{bmatrix} \quad b = \vec{0}$$

In the vector u there is a "dominant element" in position u_i where its absolute value is either equal to or larger than the absolute value any other element in u. Lets call this maximum value α.

Every element in u cannot be α. If this were the case then row i multiplied by u would not result in a 0 element for b which is needed in order for b to be the 0 vector.

$$\begin{array}{lccccccccccc}
(1) & |u_1| & = & |u_2| & = & ... & = & |u_n| & = & \alpha & & (premise) \\
(2) & A_{i1}u_1 & + & A_{i2}u_2 & + & ... & + & A_{in}u_n & = & 0 & & (row, column\ multiplication) \\
(3) & \pm A_{i1}\alpha & \pm & A_{i2}\alpha & \pm & ... & \pm & A_{in}\alpha & = & 0 & & (substitution) \\
(4) & (\pm A_{i1} & \pm & A_{i2} & \pm & ... & \pm & A_{in}) & = & 0 & & (distributivity) \\
(5) & \pm A_{i1} & \pm & A_{i2} & \pm & ... & \pm & A_{in} & = & 0 & & (multiplicative\ inverse) \\
(6) & |A_{i1}| & = & |A_{i2}| & + & ... & + & |A_{in}| & & & & (check\ for\ SDD)
\end{array}$$

(6) contradicts the premise that A_{nn} is SDD thus every entry of u_n cannot be α.

In order for b_1 to equal 0 then $\sum_{i=1}^{n} A_{1i}u_i = 0$. Because $|A_{11}| > \sum_{j \neq i} |A_{ij}|$, then position u_1 cannot be α. If u_1 cannot be α then what about position u_2? For the same reason u_1 cannot be α due to the the magnitude of A_{ii} in row 1 of A, u_2 cannot be α due to the magnitude of A_{ii} in row 2 of A. This logic then continues from u_2 until u_n. As a result no element in u can be the maximum element and all elements in u cannot be the maximum element. Therefore there is no vector u that we can create such that $Au = 0$.

If there is no u other than the 0 vector that can be created such that $Au = 0$, then A is nonsingular (Definition NM[67]), a contradiction to our premise.

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In knowing that matrix A is nonsingular provided that A is SDD we can now move on to Gershgorin's Theorem.

2 Gershgorin's Theorem

Theorem 2.1 *(Gershgorin's Theorem Round 1)*

Every eigenvalue of matrix A_{nn} satisfies:
$|\lambda - A_{ii}| \leq \sum_{j \neq i} |A_{ij}| \qquad i \in \{1, 2, ..., n\}$

Proof

□

Suppose that λ is an eigenvalue of the matrix A. The matrix $\lambda I - A$ is SDD if $|\lambda - A_{ii}| > \sum_{j \neq i} |A_{ij}|$ for every i. If Theorem 2.1 is not satisfied then $\lambda I - A$ is SDD. If $\lambda I - A$ is SDD then it is nonsingular by Theorem 1.1 and as a result λ is not an eigenvalue. If λ is to be an eigenvalue then Theorem 2.1 must hold.

■

In analyzing this theorem we see that every eigenvalue of the matrix A must be within a distance d of A_{ii} for some i. Since in general eigenvalues are elements of $\mathbb{C}$, we can visualize an eigenvalue as a point in the complex plane, where that point has to be within distance d of A_{ii} for some i. This brings us to Definition 2.

Definition 2.1 (Gershgorin's disc)

Let $d_i = \sum_{j \neq i} |A_{ij}|$. Then the set $D_i = \{z \in \mathbb{C} : |z - A_{ii} \leq d_i\}$ is called the ith Gershgorin disc of the matrix A. This disc is the interior plus the boundary of a circle. The circle has a radius d_i and is centered at (the real part of A_{ii}, the imaginary part of A_{ii})

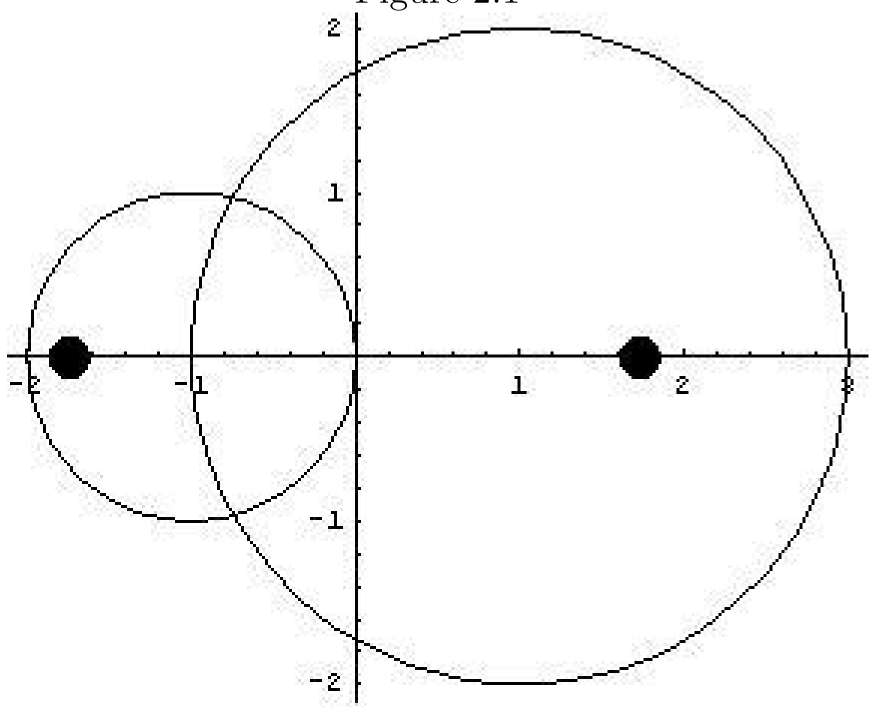
Example 2.1

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

As a result of matrix A we have eigenvalues $\sqrt{3}, -\sqrt{3}$.

From the rows of matrix A we get a disc with radius 2 centered at (1,0) and a disc of radius 1 centered at (-1,0). Plotting both the discs and the eigenvalues complex plane we get:

Figure 2.1



From Definition 2.1 we see that for the matrix A_{nn} there are n discs in the complex plane, each centered on one of the diagonal entries of the matrix A_{nn}. From Theorem 2.1 we know that every eigenvalue must lie within one of these discs. However it does not say that each disc has an eigenvalue.

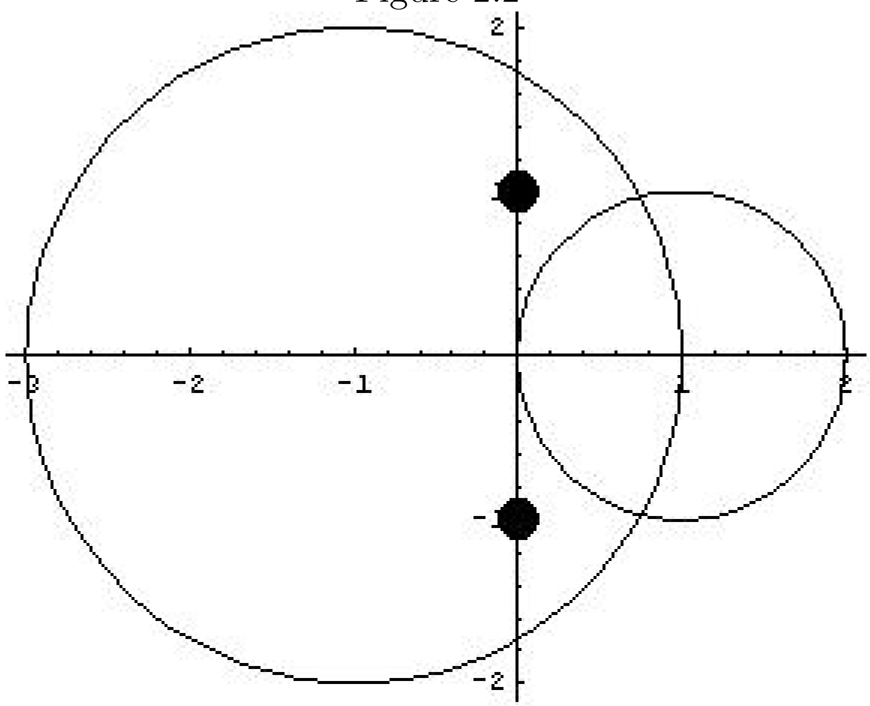
Example 2.2

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

As a result of matrix A we have eigenvalues $i, -i$.

From the rows of matrix A we get a disc with radius 1 centered at (1,0) and a disc of radius 2 centered at (-1,0). Plotting both the discs and the eigenvalues in the complex plane we get:

Figure 2.2



It is clearly visible that all of the eigenvalues fall within the disc defined by the 2nd row and none fall within the disc defined by the 1st row.

Theorem 2.2 *(Gershgorin in Respect to Columns)*

Every eigenvalue of a matrix A must lie in a Gershgorin disc corresponding to the columns of A.

Proof

□

Theorem 2.1 and the resulting definition 2 gives us Gershgorin discs that correspond to the rows of A, where A is the matrix whose eigenvalues we are looking for. If we transpose matrix A we then get the columns of matrix A as the rows of matrix A^t. As we know from Theorem ETM[421] the eigenvalues of A are the same as the eigenvalues of A^t additionally matrix A^t must also obey Theorem 2.1.

Putting this all together we have the set of eigenvalues that are in both A and A^t. Because the rows of A^t correspond to the columns of A, the eigenvalues fall inside Gershgorin discs corresponding to the the columns of A due to A^t obeying Theorem 2.1.

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Now we come to one of the most interesting properties of Gershgorin discs.

Theorem 2.3 *(Gershgorin's Theorem Round 2)*

A Subset G of the Gershgorin discs is called a disjoint group of discs if no disc in the group G intersects a disc which is not in G. If a disjoint group G contains r nonconcentric discs, then there are r eigenvalues.

Proof

□

Suppose $A \in A_{nn}$. Define A'(p) as the matrix A with the off diagonal elements multiplied by the variable p, where p is defined from 0 to 1.

At A'(0) we have Gershgorin discs with a radius of 0 centered at the location of the diagonal elements and eigenvalues equal to the diagonal elements. As p increases the Gershgorin discs of A'(p) will increase in radius based on p. Additionally the eigenvalues will also move. These movements can be tracked using the roots of the characateristic polynomial belonging to A'(p).

With the variable p found in the off diagonal elements of A'(p), the characteristic polynomial of A'(p) is a function based on two variables p and x. With the variable p defined from 0 to 1, the characteristic polynomial will be continuous from 0 to 1. If the characteristic polynomial is continous then the roots are continuous. Since the eigenvalues are found within the roots, the change that the eigenvalues experience as p

$\to 1$ is continuous. As such the eigenvalues will trace a continuous path from one point to another based on the variable p.

Theorem 2.1 tells us that an eigenvalue always has to be within a disc, and due to the continuity of the eigenvalue's path there is no way that an eigenvalue can move from one isolated group to another isolated group without being found in a region outside of any disc. Being outside of a disc violates Theorem 2.1 and therefore every disjoing group G that has n discs in it must have n eigenvalues in it.

$\blacksquare$

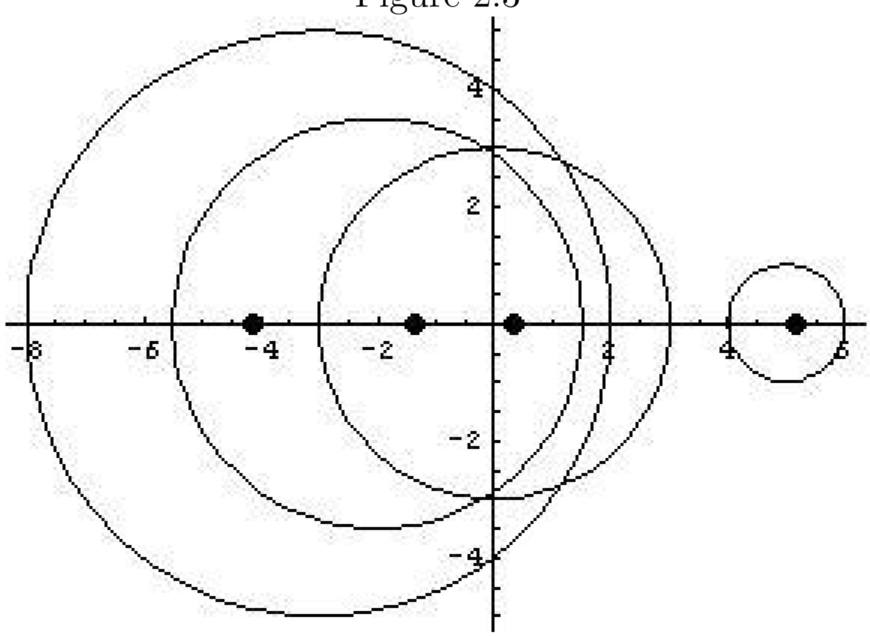
Example 2.3

$$A = \begin{bmatrix} 5 & 0 & 0 & -1 \\ 1 & 0 & -1 & 1 \\ -1.5 & 1 & -2 & 1 \\ -1 & 1 & 3 & -3 \end{bmatrix}$$

As a result of matrix A we have eigenvalues $\approx 5.17, \approx -4.15, \approx -1.38, \approx 0.35$.

From A we get a disc with radius 1 centered at (5,0), a disc of radius 3 centered at (0,0), a disc with radius 3.5 centered at (-2,0) and a disc with radius 5 centered at (-3,0). Plotting all four of these discs and the eigenvalues in the (Real, Imaginary) plane we get:

Figure 2.3



As you can see there is a Group G_1 made up of 1 disc centered at (5,0) inside this group there is 1 eigenvalue. In the larger group, G_2 there are 4 discs and inside the group there are 4 eigenvalues.

Theorem 2.4 *(Real disjoint Gershgorin Disc)*

If matrix A_{nn} has a disjoint Gershgorin Disc, P, created from a row with a real diagonal element then the eigenvalue within disc P is real.

Proof

□

Suppose $A \in A_{nn}$ λ is an eigenvalue of A_{nn} and lies within disc q created from a row which has a real diagonal element. Let q be disjoint from all other Gershgorin Discs. If λ = x+iy, where x and y are both nonzero real numbers, then another eigenvalue of A_{nn} is x-iy. Since x-iy is equally distant from the center of the disc as x+iy, it follows that x-iy is in the disc q. However this means that there are two eigenvalues inside the isolated Gershgorin Disc q given that the center of q is on the real axis. This contradicts Theorem 2.3 and therefore Theorem 2.4 must hold.

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A good example of this theorem is example 2.3

As a result of Theorems 2.1 - 2.4 ranges for eigenvalues can be found. This is especially helpful for large matrices where calculating the eigenvalues can be impractical. Additionally, Theorem 2.4 has applications to stability of dynamic systems.

References

[1] Griffel, D.H. *Linear Algebra and its Applications. Volume 1*

[2] Beezer, Robert. *A First Course in Linear Algebra*

[3] Gershgorin's Circle Theorem. *http://en.wikipedia.org*