1. Find all solutions to the system of equations below, using the inverse of the coefficient matrix. Demonstrate how to find the inverse by a procedure using row operations to obtain the extended echelon form of the matrix. (15 points)

\[
\begin{align*}
    x_1 + 3x_2 - x_3 &= -4 \\
    -x_1 - 2x_2 + 3x_3 &= 8 \\
    2x_1 + 5x_2 - 3x_3 &= -9
\end{align*}
\]

Solution: By Theorem SLEMM we can view this linear system as a vector equation of the form \(Ax = b\). Attach the \(3 \times 3\) identity matrix to the coefficient matrix \(A\) and row-reduce,

\[
\begin{bmatrix}
1 & 3 & -1 & 1 & 0 & 0 \\
-1 & -2 & 3 & 0 & 1 & 0 \\
2 & 5 & -3 & 0 & 0 & 1
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 & -9 & 4 & 7 \\
0 & 1 & 0 & 3 & -1 & -2 \\
0 & 0 & 1 & -1 & 1 & 1
\end{bmatrix}
\]

So by Theorem CINM we have the inverse of the coefficient matrix as

\[
A^{-1} = \begin{bmatrix}
-9 & 4 & 7 \\
3 & -1 & -2 \\
-1 & 1 & 1
\end{bmatrix}
\]

We recognize \(A\) as a nonsingular matrix (Theorem NI), so we can apply Theorem SNCM to obtain the unique solution to this system as the vector

\[
x = A^{-1}b = \begin{bmatrix}
-9 & 4 & 7 \\
3 & -1 & -2 \\
-1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
-4 \\
8 \\
-9
\end{bmatrix}
= \begin{bmatrix}
5 \\
-2 \\
3
\end{bmatrix}
\]
2. For the matrix \( A \) below, find a linearly independent set of vectors \( S \) whose span equals the column space of \( A \), that is \( \langle S \rangle = \mathcal{C}(A) \). The two parts of this problem ask for different versions of the set \( S \). (20 points)

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 \\
-4 & 1 & -3 & -7 \\
1 & -7 & -6 & -5 \\
\end{bmatrix}
\]

(a) Construct \( S \) so that the vectors are also columns of \( A \).

Solution: In anticipation of using Theorem BCS, we row-reduce \( A \),

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
-4 & 1 & -3 & -7 \\
1 & -7 & -6 & -5 \\
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

So the set of pivot columns is \( D = \{1, 2\} \), so Theorem BCS says we can grab the first two columns of \( A \) and this set will have the requested properties.

\[
S = \left\{ \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix} \right\}
\]

(b) Construct \( S \) by a procedure that uses the row space of the transpose of \( A \), \( \mathcal{R}(A^t) \).

Solution: By Theorem CSRST the row space of \( A^t \) will equal the column space of \( A \). We can obtain a set describing the row space of a matrix by row-reducing and keeping nonzero rows — this is Theorem BRS. These vectors will be linearly independent, and their span equals the subset, as requested.

\[
\begin{bmatrix}
1 & -4 & 1 \\
2 & 1 & -7 \\
3 & -3 & -6 \\
4 & -7 & -5 \\
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & -3 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

Writing the rows of this matrix as column vectors, we have the set

\[
S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}
\]
3. Construct the extended echelon form of the matrix $A$ below. For each of the four subsets, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathcal{C}(A)$ and $\mathcal{L}(A)$ (respectively: row space, null space, column space, left null space) find a linearly independent set of vectors whose span equals the desired subset by using information from the extended echelon form. (20 points)

$$A = \begin{bmatrix}
6 & -5 & 4 & 7 & -11 \\
-1 & 1 & -1 & -1 & 2 \\
-12 & 9 & -6 & -15 & 21 \\
4 & -3 & 2 & 5 & -7 \\
\end{bmatrix}$$

Solution: With four rows, we adjoin a $4 \times 4$ identity matrix and row-reduce to obtain the extended echelon form (Definition EEF),

$$\begin{bmatrix}
6 & -5 & 4 & 7 & -11 & 1 & 0 & 0 & 0 \\
-1 & 1 & -1 & -1 & 2 & 0 & 1 & 0 & 0 \\
-12 & 9 & -6 & -15 & 21 & 0 & 0 & 1 & 0 \\
4 & -3 & 2 & 5 & -7 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}
1 & 0 & -1 & 2 & -1 & 0 & 3 & 0 & 1 \\
0 & 1 & -2 & 1 & 1 & 0 & 4 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\
\end{bmatrix}$$

From this matrix we extract the submatrices $C$ and $L$ (as defined in Definition EEF),

$$C = \begin{bmatrix} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & -2 & 1 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Applying Theorem FS, along with the indicated theorems about null spaces and row spaces, will give us the requested sets,

$$\mathcal{N}(A) = \mathcal{N}(C) = \langle \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rangle \quad \text{Theorem BNS}$$

$$\mathcal{R}(A) = \mathcal{R}(C) = \langle \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \rangle \quad \text{Theorem BRS}$$

$$\mathcal{C}(A) = \mathcal{N}(L) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix} \rangle \quad \text{Theorem BNS}$$

$$\mathcal{L}(A) = \mathcal{R}(L) = \langle \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle \quad \text{Theorem BRS}$$
4. Give an example of a $4 \times 4$ unitary matrix that is not the identity matrix. (15 points)

Solution: the simplest route to such an example is to begin with the $4 \times 4$ identity matrix and rearrange the columns. Since $I_4$ is unitary ($I_4^* I_4 = I_4$), its columns form an orthonormal set by Theorem CUMOS. Any rearrangement of the columns will still be a matrix whose columns are an orthonormal set, and hence by Theorem CUMOS will be a unitary matrix. Here’s a concrete example of such a solution, formed by swapping columns 1 and 2 of $I_4$.

$$
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

5. Suppose that $A$ is an $m \times n$ matrix, and $B$ and $C$ are $n \times p$ matrices. Give a careful proof that $A(B + C) = AB + AC$ (i.e. do not just simply quote a theorem from the book). (15 points)

Solution: This is Theorem MMDAA. See the proof given there.

6. Suppose that $A$ is a nonsingular matrix. Prove that $(A^t)^{-1} = (A^{-1})^t$. (15 points)

Solution: This is basically Theorem MIT. However the hypothesis is that $A$ is nonsingular, so we need to first ascertain that $A^{-1}$ really exists. Theorem NI does this for us. With $A^{-1}$ in hand we can demonstrate that $(A^{-1})^t$ functions as the inverse of $A^t$ by meeting the requirements of Definition MI.