Math 290 Name: Key Dr. Beezer
Quiz SLE Spring 2007

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Find all solutions to the system of equations below. Express your solution(s) as a set. (15 points)

$$\begin{aligned}
-x_1 + 5x_2 &= -8\\
-2x_1 + 5x_2 + 5x_3 + 2x_4 &= 9\\
-3x_1 - x_2 + 3x_3 + x_4 &= 3\\
7x_1 + 6x_2 + 5x_3 + x_4 &= 30
\end{aligned}$$

Solution: We row-reduce the augmented matrix of the system of equations,

$$\begin{bmatrix} -1 & 5 & 0 & 0 & -8\\ -2 & 5 & 5 & 2 & 9\\ -3 & -1 & 3 & 1 & 3\\ 7 & 6 & 5 & 1 & 30 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 3\\ 0 & \boxed{1} & 0 & 0 & -1\\ 0 & 0 & \boxed{1} & 0 & 2\\ 0 & 0 & 0 & \boxed{1} & 5 \end{bmatrix}$$

The reduced row-echelon form of the matrix is the augmented matrix of the system $x_1 = 3$, $x_2 = -1$, $x_3 = 2$, $x_4 = 5$, which has a unique solution. As a set of column vectors, the solution set is

$$S = \left\{ \begin{bmatrix} 3\\ -1\\ 2\\ 5 \end{bmatrix} \right\}$$

2. Find all solutions to the system of equations below. Express your solution(s) as a set. (15 points)

$$\begin{aligned}
x_1 + 2x_2 - 4x_3 - x_4 &= 32\\
x_1 + 3x_2 - 7x_3 - x_5 &= 33\\
x_1 + 2x_3 - 2x_4 + 3x_5 &= 22
\end{aligned}$$

Solution: We row-reduce the augmented matrix of the system of equations,

$$\begin{bmatrix} 1 & 2 & -4 & -1 & 0 & 32\\ 1 & 3 & -7 & 0 & -1 & 33\\ 1 & 0 & 2 & -2 & 3 & 22 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 5 & 6\\ 0 & \boxed{1} & -3 & 0 & -2 & 9\\ 0 & 0 & 0 & \boxed{1} & 1 & -8 \end{bmatrix}$$

With no leading 1 in the final column, we recognize the system as consistent (Theorem RCLS). Since the system is consistent, we compute the number of free variables as $n - r = 5 - 3 = 2$ (), and we see that columns 3 and 5 are not pivot columns, so x_3 and x_5 are free variables. We convert each row of the reduced row-echelon form of the matrix into an equation, and solve it for the lone dependent variable, as in expresdsion in the two free variables.

$$\begin{aligned}
x_1 + 2x_3 + 5x_5 = 6 &\quad\rightarrow\quad x_1 = 6 - 2x_3 - 5x_5\\
x_2 - 3x_3 - 2x_5 = 9 &\quad\rightarrow\quad x_2 = 9 + 3x_3 + 2x_5\\
x_4 + x_5 = -8 &\quad\rightarrow\quad x_4 = -8 - x_5
\end{aligned}$$

These expressions give us a convenient way to describe the solution set, S.

$$S = \left\{ \left. \begin{bmatrix} 6 - 2x_3 - 5x_5 \\ 9 + 3x_3 + 2x_5 \\ x_3 \\ -8 - x_5 \\ x_5 \end{bmatrix} \right| x_3,\, x_5 \in \mathbb{C} \right\}$$

3. Find a matrix that is row-eqivalent matrix to C and in reduced row-echelon form. Perform the necessary computations by hand, not with a calculator. (15 points)

$$\begin{bmatrix} 1 & 1 & 1 \\ -4 & -3 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution: Following the algorithm of Theorem REMEF, and working to create pivot columns from left to right, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ -4 & -3 & -2 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{4R_1+R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{-1R_2+R_1}$$

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{1R_2+R_3} \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Find the null space of the matrix B below. (15 pts)

$$B = \begin{bmatrix} -6 & 4 & -36 & 6 \\ 2 & -1 & 10 & -1 \\ -3 & 2 & -18 & 3 \end{bmatrix}$$

Solution: We form the augmented matrix of the homogeneous system $\mathcal{LS}(B,\, \mathbf{0})$ and row-reduce the matrix,

$$\begin{bmatrix} -6 & 4 & -36 & 6 & 0 \\ 2 & -1 & 10 & -1 & 0 \\ -3 & 2 & -18 & 3 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & 1 & 0 \\ 0 & \boxed{1} & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We knew ahead of time that this system would be consistent (Theorem HSE), but we can now see there are $n-r = 4-2 = 2$ free variables, namely x_3 and x_4 (Theorem FVCS). Based on this analysis, we can rearrange the equations associated with each nonzero row of the reduced row-echelon form into an expression for the lone dependent variable as a function of the free variables. We arrive at the solution set to the homogeneous system, which is the null space of the matrix by Definition NSM,

$$\mathcal{N}(B) = \left\{ \left. \begin{bmatrix} -2x_3 - x_4 \\ 6x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \right| x_3,\, x_4 \in \mathbb{C} \right\}$$

5. Let A be the coefficient matrix of the system of equations in problem #1. Is A nonsingular or singular? Suppose you had not already finished the first problem — explain what you could infer about the solution set for problem #1 based only on what you have learned here about A being singular or nonsingular. (15 points)

   Solution: We could row-reduce the coefficient matrix of the system, and from the work done in the solution to problem #1 we see that we would arrive at the 4×4 identity matrix, I_4 (Definition IM). By Theorem NRRI, we know the coefficient matrix is nonsingular. According to Theorem NMUS we know that the system was guaranteed in advance to have a unique solution, based only on the extra information that the coefficient matrix is nonsingular.

6. Say **as much as possible** about solutions to the systems described below. Give concrete reasons for your conclusions. (15 points)
   (a) Homogeneous with 10 equations and 13 variables.

   Solution: $13 > 10$, so Theorem HMVEI tells us there will be infinitely many solutions. The proof of Theorem HSE tells us that one of these solutions is the trivial solution.

   (b) 6 equations in 12 variables.

   Solution: The system could be inconsistent. If it is consistent, and since $12 > 6$, then Theorem CMVEI says we will have infinitely many solutions. So there are two possibilities. Theorem PSSLS allows to state equivalently that a unique solution is an impossibility.

   (c) 8 equations, 6 variables. Reduced row-echelon form of the augmented matrix of the system has 7 pivot coulmns.

   Solution: 7 pivot columns implies that there are $r = 7$ nonzero rows (so row 8 is all zeros in the reduced row-echelon form). Then $n + 1 = 6 + 1 = 7 = r$ and Theorem ISRN allows to conclude that the system is inconsistent.

7. Suppose that the coefficient matrix of a system of linear equations has two columns that are identical. Prove that the system has infinitely many solutions. (10 points)

   Solution: Since the system is consistent, we know there is either a unique solution, or infinitely many solutions (Theorem PSSLS). If we perform row operations (Definition RO) on the augmented matrix of the system, the two equal columns of the coefficient matrix will suffer the same fate, and remain equal in the final reduced row-echelon form. Suppose both of these columns are pivot columns (Definition RREF). Then there is single row containing the two leading 1's of the two pivot columns, a violation of reduced row-echelon form (acronymrefdefinitionRREF). So at least one of these columns is not a pivot column, and the column index indicates a free variable in the description of the solution set (Definition IDV). With a free variable, we arrive at an infinite solution set (Theorem FVCS).