Math 290 Name: Key Dr. Beezer
Quiz R Fall 2007

Show *all* of your work and *explain* your answers fully. There is a total of 120 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. P_2 is the vector space of polynomials with degree at most 2, and M_{22} is the vector space of 2×2 matrices. Consider the linear transformation $T\colon P_2 \mapsto M_{22}$ defined below. (45 points)

$$T\left(a + bx + cx^2\right) = \begin{bmatrix} -b+3c & a+3b-7c \\ a+b-c & a+2b-4c \end{bmatrix}$$

(a) Determine the matrix representation of T relative to the bases below.

$$B = \left\{1,\ 1+x,\ 1+x+x^2\right\} \qquad C = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right\}$$

Solution: Applying Definition MR,

$$\rho_C\left(T\left(1\right)\right) = \rho_C\left(\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\right) = \rho_C\left((-1)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 0\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + 0\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} + 1\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right) = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

$$\rho_C\left(T\left(1+x\right)\right) = \rho_C\left(\begin{bmatrix}-1 & 4\\2 & 3\end{bmatrix}\right) = \rho_C\left((-5)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 2\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + (-1)\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} + 3\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right) = \begin{bmatrix}-5\\2\\-1\\3\end{bmatrix}$$

$$\rho_C\left(T\left(1+x+x^2\right)\right) = \rho_C\left(\begin{bmatrix}2 & -3\\1 & -1\end{bmatrix}\right) = \rho_C\left(5\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + (-4)\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + (2)\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} + (-1)\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right) = \begin{bmatrix}5\\-4\\2\\-1\end{bmatrix}$$

So the resulting matrix representation is

$$M^T_{B,C} = \begin{bmatrix} -1 & -5 & 5\\ 0 & 2 & -4\\ 0 & -1 & 2\\ 1 & 3 & -1 \end{bmatrix}$$

(b) Use your matrix representation from part (a) to determine the kernel of T, $\mathcal{K}(T)$.

Solution: The statement of Theorem KNSI tells us that the kernel of the linear transormation is isomorphic to the null space of the matrix representation, and the proof of this result tells us that an isomorphism is ρ_B (or its inverse, depending on the direction). So we compute the null space of the matrix representation with Theorem BNS,

$$\begin{bmatrix} -1 & -5 & 5\\ 0 & 2 & -4\\ 0 & -1 & 2\\ 1 & 3 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 5\\ 0 & \boxed{1} & -2\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}$$

So

$$\mathcal{N}\left(M^T_{B,C}\right) = \left\langle\left\{\begin{bmatrix}-5\\2\\1\end{bmatrix}\right\}\right\rangle$$

To find the kernel of the linear transformation, we want to un-coordinatize the column vector that is the basis element of the null space of the matrix representation, that is, we apply the inverse of ρ_B,

$$\rho_B^{-1}\left(\begin{bmatrix}-5\\2\\1\end{bmatrix}\right) = -5\,(1) + 2\,(1+x) + 1\left(1+x+x^2\right) = -2+3x+x^2$$

$$\mathcal{K}(T) = \left\langle\left\{-2+3x+x^2\right\}\right\rangle$$

(c) Use your matrix representation from part (a) to determine the range of T, $\mathcal{R}(T)$.

Solution: The statement of Theorem RCSI tells us that the rane of the linear transormation is isomorphic to the column space of the matrix representation, and the proof of this result tells us that an isomorphism is ρ_C (or its inverse, depending on the direction). So we compute the column space of the matrix representation with Theorem CSRST and Theorem BRS,

$$\begin{bmatrix}-1&0&0&1\\-5&2&-1&3\\5&-4&2&-1\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}1&0&0&-1\\0&1&-\frac{1}{2}&-1\\0&0&0&0\end{bmatrix}$$

So

$$\mathcal{N}\left(M^T_{B,C}\right) = \left\langle\left\{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\-\frac{1}{2}\\-1\end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix}1\\0\\0\\-1\end{bmatrix},\begin{bmatrix}0\\2\\-1\\-2\end{bmatrix}\right\}\right\rangle$$

To find the kernel of the linear transformation, we want to un-coordinatize the column vector that is the basis element of the null space of the matrix representation, that is, we apply the inverse of ρ_B,

$$\rho_C^{-1}\left(\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\right) = 1\begin{bmatrix}1&0\\0&0\end{bmatrix} + 0\begin{bmatrix}1&1\\0&0\end{bmatrix} + 0\begin{bmatrix}1&1\\1&0\end{bmatrix} + -1\begin{bmatrix}1&1\\1&1\end{bmatrix} = \begin{bmatrix}0&-1\\-1&-1\end{bmatrix}$$

$$\rho_C^{-1}\left(\begin{bmatrix}0\\2\\-1\\-2\end{bmatrix}\right) = 0\begin{bmatrix}1&0\\0&0\end{bmatrix} + 2\begin{bmatrix}1&1\\0&0\end{bmatrix} + (-1)\begin{bmatrix}1&1\\1&0\end{bmatrix} + (-2)\begin{bmatrix}1&1\\1&1\end{bmatrix} = \begin{bmatrix}-1&-1\\-3&-2\end{bmatrix}$$

$$\mathcal{R}(T) = \left\langle\left\{\begin{bmatrix}0&-1\\-1&-1\end{bmatrix},\begin{bmatrix}-1&-1\\-3&-2\end{bmatrix}\right\}\right\rangle$$

2. P_2 is the vector space of polynomials with degree at most 2 Consider the linear transformation $S\colon P_2 \mapsto P_2$ defined below. (45 points)

$$S\left(a+bx+cx^2\right) = (-2a+2b+2c) + (-3a+3b+2c)\,x + (-3a+2b+3c)\,x^2$$

(a) Determine a basis for each eigenspace of S.

Solution: We need a matrix representation to work with, and we are free to choose any basis of P_2 we please. We'll choose a basis that makes the formulation of the matrix representation almost transparent, like

$B = \{1,\, x,\, x^2\}$. First, we form the matrix representation.

$$\rho_B(S(1)) = \rho_B\left(-2 - 3x - 3x^2\right) = \begin{bmatrix} -2 \\ -3 \\ -3 \end{bmatrix}$$

$$\rho_B(S(x)) = \rho_B\left(2 + 3x + 2x^2\right) = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

$$\rho_B\left(S\left(x^2\right)\right) = \rho_B\left(2 + 2x + 3x^2\right) = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

So the resulting matrix representation is

$$M = M^S_{B,B} = \begin{bmatrix} -2 & 2 & 2 \\ -3 & 3 & 2 \\ -3 & 2 & 3 \end{bmatrix}$$

Now we compute the eigenvalues and eigenspaces of the matrix representation with the techniques of Chapter E. The characteristic polynomial is $p_M(x) = -(x-2)(x-1)^2$ and the eigenspaces are

$$\mathcal{E}_M(2) = \left\langle\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$$\mathcal{E}_M(1) = \left\langle\left\{\begin{bmatrix} \frac{2}{3} \\ 0 \\ 1 \end{bmatrix},\, \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix},\, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}\right\}\right\rangle$$

For eigenspaces of the linear transformation S we uncoordinatize the basis elements, viewed as vector representations relative to B,

$$\mathcal{E}_S(2) = \left\langle\left\{1 + x + x^2\right\}\right\rangle$$
$$\mathcal{E}_S(1) = \left\langle\left\{2 + 3x^2,\, 2 + 3x\right\}\right\rangle$$

(b) Determine a basis, C, for P_2 so that a matrix representation of S relative to C will be a diagonal matrix. Explain carefully how you know that C is indeed a basis. Give the resulting matrix representation of S relative to C.

Solution: From part (a) we can see that the algebraic multiplicitity of each eigenvalue is equal to the geometric multiplicity. So by Theorem DMFE, the three column vectors that are basis elements for the matrix representation of S would provide a basis of $\mathbb{C}^3$ that would diagonalize the matrix M. The uncoordinatized versions will then be a basis (by Theorem CLI and Theorem CSS), and a matrix representation will be a diagonal matrix with the eigenvalues on the diagonal (in the proper order). So we have

$$C = \left\{1 + x + x^2,\, 2 + 3x^2,\, 2 + 3x\right\}$$

$$M^S_{C,C} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. P_2 is the vector space of polynomials with degree at most 2, and S_{22} is the vector space of 2×2 *symmetric* matrices. Consider the linear transformation $Q\colon P_2 \mapsto S_{22}$ defined below. (Hint: $\dim(S_{22}) = 3$.) (30 points)

$$Q\left(a + bx + cx^2\right) = \begin{bmatrix} 2a - b & 2a - b + c \\ 2a - b + c & a - b - c \end{bmatrix}$$

(a) Determine that Q is invertible by examining properties of a matrix representation of Q.

Solution: We are free to use any bases we wish in the construction of the matrix representation, so choose them to be as easy to work with as possible, say

$$B = \{1,\, x,\, x^2\} \qquad\qquad C = \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\, \begin{bmatrix}0&1\\1&0\end{bmatrix},\, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$

The resulting matrix representation is

$$\rho_C\left(Q\left(1\right)\right) = \rho_C\left(\begin{bmatrix}2&2\\2&1\end{bmatrix}\right) = \rho_C\left(2\begin{bmatrix}1&0\\0&0\end{bmatrix} + 2\begin{bmatrix}0&1\\1&0\end{bmatrix} + 1\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) = \begin{bmatrix}2\\2\\1\end{bmatrix}$$

$$\rho_C\left(Q\left(x\right)\right) = \rho_C\left(\begin{bmatrix}-1&-1\\-1&-1\end{bmatrix}\right) = \rho_C\left((-1)\begin{bmatrix}1&0\\0&0\end{bmatrix} + (-1)\begin{bmatrix}0&1\\1&0\end{bmatrix} + (-1)\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}$$

$$\rho_C\left(Q\left(x^2\right)\right) = \rho_C\left(\begin{bmatrix}0&1\\1&-1\end{bmatrix}\right) = \rho_C\left(0\begin{bmatrix}1&0\\0&0\end{bmatrix} + 1\begin{bmatrix}0&1\\1&0\end{bmatrix} + (-1)\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$M^{Q}_{B,C} = \begin{bmatrix}2&-1&0\\2&-1&1\\1&-1&-1\end{bmatrix}$$

We'll start our analysis with the null space of this matrix representation,

$$\begin{bmatrix}2&-1&0\\2&-1&1\\1&-1&-1\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1}&0&0\\0&\boxed{1}&0\\0&0&\boxed{1}\end{bmatrix}$$

So we see that the matrix representation is nonsingular (Theorem NMRRI). By Theorem NME9 the null space is trivial and the column space is $\mathbb{C}^3$. Then Theorem KNSI tells us the kernel of Q is trivial, and so by Theorem KILT, Q is injective. And Theorem RCSI tells us that the range of Q is all of S_{22} and so by Theorem RSLT, Q is surjective. Finally, by Theorem ILTIS, Q is invertible.

(b) Use a matrix representation to construct a "formula" for the inverse of Q, Q^{-1}.

Solution: Basically, the inverse linear transformation has a matrix representation that is the matrix inverse of the matrix representation. Here's how we determine a formula for $Q^{-1}\colon S_{22} \mapsto P_2$,

$$\begin{aligned}
Q^{-1}\left(\begin{bmatrix}a&b\\b&c\end{bmatrix}\right) &= \rho_B^{-1}\left(M^{Q^{-1}}_{C,B}\,\rho_C\left(\begin{bmatrix}a&b\\b&c\end{bmatrix}\right)\right) && \text{Theorem FTMR}\\
&= \rho_B^{-1}\left(\left(M^{Q}_{B,C}\right)^{-1}\rho_C\left(\begin{bmatrix}a&b\\b&c\end{bmatrix}\right)\right) && \text{Theorem IMR}\\
&= \rho_B^{-1}\left(\left(M^{Q}_{B,C}\right)^{-1}\begin{bmatrix}a\\b\\c\end{bmatrix}\right) && \text{Definition VR}\\
&= \rho_B^{-1}\left(\begin{bmatrix}2&-1&-1\\3&-2&-2\\-1&1&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}\right) && \text{Theorem CINM}\\
&= \rho_B^{-1}\left(\begin{bmatrix}2a-b-c\\3a-2b-2c\\-a+b\end{bmatrix}\right) && \text{Definition MVP}\\
&= (2a-b-c) + (3a-2b-2c)\,x + (-a+b)\,x^2 && \text{Definition VR}
\end{aligned}$$