Math 290 Name: Key Dr. Beezer
Quiz D&E Fall 2007

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Given the matrix $A$ below, find a row-equivalent matrix $B$ in reduced row-echelon form. Then find a matrix $C$ such that $B = CA$. (15 points)

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

Solution: We find the requested matrix $B$ by using row operations (Definition RO) to arrive at a matrix in reduced row-echelon form (Definition RREF),

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix} \xrightarrow{-1R_1} \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 2 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{1R_2+R_1} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}$$

From Theorem EMDRO we know that we can effect row operations with matrix multiplication by elementary matrices, so

$$B = E_{2,1}(1)\, E_{1,2}(-2)\, E_1(-1)\, A$$

so

$$\begin{aligned} C &= E_{2,1}(1)\, E_{1,2}(-2)\, E_1(-1) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

A simple check of the product $CA$ shows that $C$ converts $A$ into reduced row-echelon form.

2. Without using a calculator, find an eigenvector of the matrix $F$. (15 points)

$$F = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

Solution: Before we can find an eigenvector, we need to know an eigenvalue of $F$, so we will compute the characteristic polynomial of $F$ (Definition CP),

$$\begin{aligned} p_F(x) &= \det(F - xI_2) && \text{Definition CP} \\ &= \begin{vmatrix} -x & -2 \\ 2 & -x \end{vmatrix} \\ &= (-x)(-x) - (-2)(2) && \text{Theorem DMST} \\ &= x^2 + 4 = (x - 2i)(x + 2i) \end{aligned}$$

Eigenvalues are the roots of the characteristic polynomial (Theorem EMRCP), so $\lambda = 2i$ and $\lambda = -2i$ are the eigenvalues of $F$. We'll run with just one of these, choosing $\lambda = 2i$, and computing the eigenspace of $F - (2i)I_2$,

$$F - (2i)I_2 = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -i \\ 0 & 0 \end{bmatrix}$$

So, applying Theorem BNS, we have the eigenspace of $\lambda = 2i$,

$$\mathcal{E}_F(2i) = \mathcal{N}(F - (2i)I_2) = \left\langle\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}\right\rangle$$

Any nonzero scalar multiple of this basis vector will qualify as an eigenvector of $F$, whereas if we had chosen to work with $\lambda = -2i$, eigenvectors would be nonzero multiples of $\mathbf{x} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

3. Let $B$ be the $4 \times 4$ matrix below. (40 points)

$$B = \begin{bmatrix} -28 & 9 & -63 & -9 \\ -6 & -7 & -18 & -18 \\ 12 & -6 & 26 & 0 \\ 0 & 6 & 3 & 11 \end{bmatrix}$$

(a) Use a calculator to find the eigenvalues of $B$, along with their algebraic multiplicities. (Input check: the eigenvalues are all integers.) Then write the characteristic polynomial of $B$, $p_B(x)$.

Solution: You should find two distinct eigenvalues $\lambda = 2$ and $\lambda = -1$, each with algebraic multiplicity 2. Since eigenvalues are roots of the characteristic polynomial (Theorem EMRCP), we know that

$$p_B(x) = (x-2)^2(x-(-1))^2 = x^4 - 2x^3 - 3x^2 + 4x + 4$$

(b) Find a basis for each of the eigenspaces of $B$ and state the geometric multiplicity of each eigenvalue of $B$.

Solution: Eigenspaces can be described as null spaces (Theorem EMNS) and we can use Theorem BNS to arrive at basis vectors,

$$\lambda = 2 \qquad B - 2I_4 = \begin{bmatrix} -30 & 9 & -63 & -9 \\ -6 & -9 & -18 & -18 \\ 12 & -6 & 24 & 0 \\ 0 & 6 & 3 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{9}{4} & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_B(2) = \mathcal{N}(B - 2I_4) = \left\langle\left\{\begin{bmatrix} -\frac{9}{4} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{4} \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$$= \left\langle\left\{\begin{bmatrix} -9 \\ -2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \\ 4 \end{bmatrix}\right\}\right\rangle$$

$$\lambda = -1 \qquad B - (-1)I_4 = \begin{bmatrix} -27 & 9 & -63 & -9 \\ -6 & -6 & -18 & -18 \\ 12 & -6 & 27 & 0 \\ 0 & 6 & 3 & 12 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{5}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_B(-1) = \mathcal{N}(B + I_4) = \left\langle \left\{ \begin{bmatrix} -\frac{5}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$$= \left\langle \left\{ \begin{bmatrix} -5 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

Each eigenspace has a basis of size 2, so the geometric multiplicity of each eigenvalue is 2.

(c) Based just on your answers to (a) and (b), describe a theorem (acronym, or summarize) that allows you to now conclude that $B$ is diagonalizable. Find matrices $S$ and $D$ such that $D$ is a diagonal matrix and $D = S^{-1}BS$.

Solution: Since the algebraic multiplicty of each eigenvalue is equal to the geometric multiplicity of the eigenvalue, Theorem DMFE applies, and we know $B$ is diagonalizable. The proof of Theorem DMFE says we can grab a basis of each eigenspace and throw them together in one big set, which will provide the 4 linearly independent eigenvectors of Theorem DC. So we can state (and we do not have to verify any computations, except as a check), that the required matrix $S$ has columns that are eigenvectors and the diagonal matrix has the corresponding eigenvalues on the diagonal,

$$S = \begin{bmatrix} -9 & -3 & -5 & -1 \\ -2 & -6 & -1 & -2 \\ 4 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

4. Suppose that $A$ is a nonsingular matrix, and $\lambda$ is an eigenvalue of $A$. Prove that $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$. (15 points)

Solution: This is Theorem EIM, see the proof given there.

5. Suppose that $A$ and $B$ are square matrices of the same size and $A$ is similar to $B$. Prove that $A$ and $B$ have equal determinants, $\det(A) = \det(B)$. (15 points)

Solution: Since $A$ and $B$ are similar, there exists a nonsingular matrix $S$ such that $A = S^{-1}BS$. Then

$$\begin{aligned}
\det(A) &= \det\left(S^{-1}BS\right) && \text{Definition SIM} \\
&= \det\left(S^{-1}\right)\det(B)\det(S) && \text{Theorem DRMM} \\
&= \det\left(S^{-1}\right)\det(S)\det(B) && \text{Property MCCN} \\
&= \det\left(S^{-1}S\right)\det(B) && \text{Theorem DRMM} \\
&= \det(I)\det(B) && \text{Definitiom MI} \\
&= 1\det(B) && \text{Theorem DIM} \\
&= \det(B) && \text{Property OCN}
\end{aligned}$$