Math 290 Name: Key Dr. Beezer
Quiz V Fall 2007

Show *all* of your work and *explain* your answers fully. There is a total of 105 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Determine if the vector $\mathbf{y}$ is an element of the span of S, $\langle S\rangle$. (15 points)

$$\mathbf{y} = \begin{bmatrix}-1\\5\\2\end{bmatrix} \qquad S = \left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\ \begin{bmatrix}-1\\5\\2\end{bmatrix},\ \begin{bmatrix}3\\5\\6\end{bmatrix},\ \begin{bmatrix}5\\2\\5\end{bmatrix}\right\}$$

Solution: By Definition SSCV we want to know if there are scalars a_1, a_2, a_3, a_4 such that

$$a_1\begin{bmatrix}1\\2\\1\end{bmatrix} + a_2\begin{bmatrix}-1\\5\\2\end{bmatrix} + a_3\begin{bmatrix}3\\5\\6\end{bmatrix} + a_4\begin{bmatrix}5\\2\\5\end{bmatrix} = \begin{bmatrix}-1\\5\\2\end{bmatrix}$$

By Theorem SLSLC we can replace the quest for solutions to this vector equality with a system of equations. We display the augmented matrix of this system and row-reduce this matrix to investigate solutions of the system,

$$\begin{bmatrix}1 & -1 & 3 & 5 & -1\\ 2 & 5 & 5 & 2 & 5\\ 1 & 2 & 6 & 5 & 2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0 & 1 & 0\\ 0 & \boxed{1} & 0 & -1 & 1\\ 0 & 0 & \boxed{1} & 1 & 0\end{bmatrix}$$

With no leading 1 in the final column, Theorem RCLS tells us the system is consistent. The existence of a solution then tells us that $\mathbf{y}$ is an element of the span of S, $\mathbf{y} \in \langle S\rangle$.

2. Find a set of column vectors, T, such that (1) the span of T is the null pace of B, $\langle T\rangle = \mathcal{N}(B)$ and (2) T is a linearly independent set. (15 points)

$$B = \begin{bmatrix}2 & 1 & 1 & 1\\ -4 & -3 & 1 & -7\\ 1 & 1 & -1 & 3\end{bmatrix}$$

Solution: The conclusion of Theorem BNS gives us everything this question asks for. We need the reduced row-echelon form of the matrix so we can determine the number of vectors in T, and their entries.

$$\begin{bmatrix}2 & 1 & 1 & 1\\ -4 & -3 & 1 & -7\\ 1 & 1 & -1 & 3\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 2 & -2\\ 0 & \boxed{1} & -3 & 5\\ 0 & 0 & 0 & 0\end{bmatrix}$$

We can build the set T in immediately via Theorem BNS, but we will illustrate its construction in two steps. Since $F = \{3, 4\}$, we will have two vectors and can distribute strategically placed ones, and many zeros. Then we distribute the negatives of the appropriate entries of the non-pivot columns of the reduced row-echelon matrix.

$$T = \left\{\begin{bmatrix}\\ \\1\\0\end{bmatrix},\ \begin{bmatrix}\\ \\0\\1\end{bmatrix}\right\} \qquad T = \left\{\begin{bmatrix}-2\\3\\1\\0\end{bmatrix},\ \begin{bmatrix}2\\-5\\0\\1\end{bmatrix}\right\}$$

3. Determine if the sets of column vectors below are linearly independent. (30 points)

(a) $S = \left\{ \begin{bmatrix} -2 \\ 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -1 \\ 5 \end{bmatrix} \right\}$

Solution: Theorem LIVRN suggests we can answer this question by analyzing the matrix whose columns are the vectors of S. Here is the matrix and its reduced row-echelon form,

$$\begin{bmatrix} -2 & 0 & 4 \\ 1 & 1 & 6 \\ 5 & -2 & -1 \\ 6 & 6 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

The number of vectors is $n = 3$ and from the row-reduced matrix we see that the number of non-zero rows is $r = 3$, so $n = r$ and Theorem LIVRN tells us that S is linearly independent.

(b) $T = \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} \right\}$

Solution: We have $n = 4$ vectors from $\mathbb{C}^3$, so $m = 3$ and $n > m$. Thus by Theorem MVSLD, this set of vectors is linearly dependent.

4. Let W be the span of the set of vectors S below, $W = \langle S \rangle$. Find a set T so that 1) the span of T is W, $\langle T \rangle = W$, (2) T is a linearly independent set, and (3) T is a subset of S. (15 points)

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Solution: This is a straight setup for the conclusion of Theorem BS. The hypotheses of this theorem tell us to pack the vectors of W into the columns of a matrix and row-reduce,

$$\begin{bmatrix} 1 & 2 & 4 & 3 & 3 \\ 2 & -3 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

Pivot columns have indices $D = \{1, 2, 4\}$. Theorem BS tells us to form T with columns 1, 2 and 4 of S,

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

5. The three parts of this question are closely related. Most of the points available on this problem are for part (c). (15 points)

(a) Determine if $\mathbf{w}$ is an element of the span of R, $\langle R\rangle$.

$$\mathbf{w} = \begin{bmatrix} 0 \\ -6 \\ 6 \end{bmatrix} \qquad R = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \right\}$$

(b) Determine if the linear system $\mathcal{LS}(A,\, \mathbf{w})$ is consistent.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 0 \\ -6 \\ 6 \end{bmatrix}$$

(c) Discuss thoughtfully the connections you can make between parts (a) and (b) of this question.

Solution: Suppose in part (a) that we write the vector $\mathbf{w}$ as an arbitrary linear combination of the vectors in R. Then, by an application of Theorem SLSLC we can convert the question of whether or not such scalars exist into a question about solutions to a linear system of equations. The resulting system is exactly the system in part (b). In this example it happens that $\mathbf{y} \in \langle R\rangle$ and the system $\mathcal{LS}(A,\, \mathbf{w})$ is consistent.

More generally, consistent systems are those where the vector of constants is in the span of the columns of the coefficient matrix, whereas inconsistent systems are those where the vector of constants is not in the span of the columns of the coefficient matrix.

6. Suppose that $\alpha \in \mathbb{C}$, $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Prove that $\alpha\,(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$. (Note: this is Property DVAC of Theorem VSPCV, so you are being asked to do more than just quote this result from the book.) (15 points)

Solution: This result says that two vectors are equal, so we must appeal to Definition CVE and look at each entry of these vectors in general,

$$\begin{aligned}
[\alpha\,(\mathbf{x} + \mathbf{y})]_i &= \alpha\,[\mathbf{x} + \mathbf{y}]_i && \text{Definition CVSM} \\
&= \alpha\,([\mathbf{x}]_i + [\mathbf{y}]_i) && \text{Definition CVA} \\
&= \alpha\,[\mathbf{x}]_i + \alpha\,[\mathbf{y}]_i && \text{Property DCN} \\
&= [\alpha\mathbf{x}]_i + [\alpha\mathbf{y}]_i && \text{Definition CVSM} \\
&= [\alpha\mathbf{x} + \alpha\mathbf{y}]_i && \text{Definition CVA}
\end{aligned}$$

Since the two vectors agree in every entry, by Definition CVE, we say $\alpha\,(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$.