Show all of your work and explain your answers fully. There is a total of 100 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. For the linear transformation $S: \mathbb{C}^2 \rightarrow P_2$, compute the matrix representation relative to the bases $B$ and $C$ as given. In each case demonstrate the Fundamental Theorem of Matrix Representations by using the representation to compute $S\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$. (35 points)

$$S\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (-a + 3b) + (2a + b) x + (a + 3b) x^2$$

(a) $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $C=\{1, x, x^2\}$

Solution: Apply $S$ to each vector of $B$ and form a vector representation relative to $C$,

$$\rho_C\left(S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right) = \rho_C\left(-1 + 2x + x^2\right) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\rho_C\left(S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right) = \rho_C\left(3 + x + 3x^2\right) = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

Form a matrix with these vector representations as the columns, according to Definition MR,

$$M_{B,C}^S = \begin{bmatrix} -1 & 3 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Demonstrating Theorem FTMR, we see

$$S\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \rho_C^{-1}\left(\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}\right)$$

$$= \rho_C^{-1}\left(M_{B,C}^S \rho_B\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)\right)$$

$$= \rho_C^{-1}\left(M_{B,C}^S \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$$

$$= \rho_C^{-1}\begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix}$$

$$= -5 + 3x - x^2$$
(b) \( B = \left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\} \quad C = \{9 + 8x - 3x^2, 3 + 3x - x^2, -2 - 2x + x^2\} \)

Solution: Apply \( S \) to each vector of \( B \) and form a vector representation relative to \( C \),

\[
\rho_C \left( S \left( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \right) = \rho_C \left( 7x + 6x^2 \right) = \rho_C \left( -7(9 + 8x - 3x^2) + 33(3 + 3x - x^2) + 18(-2 - 2x + x^2) \right) = \begin{bmatrix} -7 \\ 33 \\ 18 \end{bmatrix}
\]

\[
\rho_C \left( S \left( \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right) \right) = \rho_C \left( 1 + 12x + 11x^2 \right) = \rho_C \left( -11(9 + 8x - 3x^2) + 56(3 + 3x - x^2) + 34(-2 - 2x + x^2) \right) = \begin{bmatrix} -11 \\ 56 \\ 34 \end{bmatrix}
\]

Form a matrix with these vector representations as the columns, according to Definition MR,

\[
M_{B,C}^S = \begin{bmatrix} -7 & -11 \\ 33 & 56 \\ 18 & 34 \end{bmatrix}
\]

Demonstrating Theorem FTMR, we see

\[
S \left( \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right) = \rho_C^{-1} \left( M_{B,C}^S \rho_B \left( \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right) \right) = \rho_C^{-1} \left( M_{B,C}^S \begin{bmatrix} 9 \\ -5 \end{bmatrix} \right) = \rho_C^{-1} \left( \begin{bmatrix} -8 \\ 17 \\ -8 \end{bmatrix} \right) = -5 + 3x - x^2
\]

2. Consider the linear transformation \( T : P_2 \rightarrow M_{22} \) defined below. Find the kernel of \( T, \mathcal{K}(T) \). (15 points)

\[
T \left( a + bx + cx^2 \right) = \begin{bmatrix} a + 2b + 3c \\ -a + b \\ 3a + b + 4c \\ 2a + 2c \end{bmatrix}
\]

Solution: Begin with a matrix representation of \( T \), relative to the nicest possible bases, say

\[
B = \{1, x, x^2\} \quad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

which yields

\[
M_{B,C}^T = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 3 & 1 & 4 \\ 2 & 0 & 2 \end{bmatrix}
\]
The kernel of $T$ is isomorphic to the null space of $M_{B,C}^T$ via the “uncoordinatization” linear transformation $\rho_B^{-1}$ (Theorem KNSI), so we first compute the null space of the matrix representation, row-reducing the matrix and using Theorem BNS,

$$N(M_{B,C}^T) = \left\langle \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

Applying $\rho_B^{-1}$ to the lone basis vector creates the polynomial $-1 - x + x^2$, which we can use as the basis vector for the kernel,

$$K(T) = \langle \{ -1 - x + x^2 \} \rangle$$

3. Find a basis for $M_{22}$ so that the linear transformation $R$ below has a diagonal matrix representation. (20 points)

$$R: M_{22} \mapsto M_{22}, \quad R \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} -17a + 12b - 36c + 6d & -6a + 4b - 12c \\ 7a - 5b + 15c - 3d & a + 2c + 2d \end{bmatrix}$$

Solution: Build a matrix representation of $R$. We can use any basis, so use the simplest possible, such as

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Then the matrix representation is

$$A = M_{B,B}^R = \begin{bmatrix} -17 & 12 & -36 & 6 \\ -6 & 4 & -12 & 0 \\ 7 & -5 & 15 & -3 \\ 1 & 0 & 2 & 2 \end{bmatrix}$$

and the eigenvalues of $M_{B,B}^R$ will be the eigenvalues of $R$. Similarly, we can extract the eigenvectors of $R$ from the eigenvectors of $M_{B,B}^R$ (Theorem EER). Using techniques from Chapter E we find the eigenspaces,

$$E_A(2) = \left\langle \left\{ \begin{bmatrix} 6 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\} \right\rangle \quad E_A(1) = \left\langle \left\{ \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \right\rangle \quad E_A(0) = \left\langle \left\{ \begin{bmatrix} -6 \\ -3 \\ 2 \\ 1 \end{bmatrix} \right\} \right\rangle$$

With algebraic multiplicities equal to geometric multiplicities, we can combine bases for each eigenspace to arrive at a basis for $\mathbb{C}^4$ (Theorem DMFE). In turn, we can “un-coordinatize” each of these eigenvectors for $A$ to arrive at an eigenvector of $R$. The four basis vectors above become the basis

$$C = \left\{ \begin{bmatrix} 6 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Though not requested, the resulting diagonal matrix representation of $R$, relative to $C$, is then

$$M_{C,C}^R = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
4. A linear transformation $T: V \mapsto V$ is called **nilpotent** if there is a positive integer $d$ such that $T^d(v) = 0_V$ for all $v \in V$. (The power $d$ on $T$ indicates repeated compositions.) Show that the linear transformation $T: P_2 \mapsto P_2$ below is nilpotent (where $P_2$ is the vector space of polynomials with degree at most 2). (15 points)

$$T(a + bx + c^2) = (-14a - 18b - 4c) + (9a + 12b + 2c)x + (7a + 9b + 2c)x^2$$

**Solution:** You might be inclined to repeatedly compose $T$ with itself, even though this gets tedious and error-prone very quickly. Instead, build a matrix representation and replace composition by matrix multiplication (Theorem MRCLT). Relative to the standard basis $B = \{1, x, x^2\}$ we easily obtain the representation,

$$P = M^T_{B,B} = \begin{bmatrix}
-14 & -18 & -4 \\
9 & 12 & 2 \\
7 & 9 & 2
\end{bmatrix}$$

Successive powers yield,

$$P^2 = \begin{bmatrix}
6 & 0 & 12 \\
-4 & 0 & -8 \\
-3 & 0 & -6
\end{bmatrix}, \quad P^3 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$$

Since $P^3 = 0$, $T^3(p(x)) = 0 + 0x + 0x^2 = 0$ for all $p(x) \in P_2$, and we see that $T$ is nilpotent.

5. Prove that the only eigenvalue of a nilpotent linear transformation is zero. (See the previous problem for the definition of a nilpotent linear transformation.) (15 points)

**Solution:** Let $x$ be an eigenvector of a linear transformation $T$ for the eigenvalue $\lambda$, and suppose that $T$ is nilpotent with index $p$. Then

$$0 = T^p(x)$$
$$= T^{p-1}(T(x))$$
$$= T^{p-1}(\lambda x)$$
$$= \lambda T^{p-1}(x)$$
$$= \lambda T^{p-2}(T(x))$$
$$= \lambda T^{p-2}(\lambda x)$$
$$= \lambda^2 T^{p-2}(x)$$
$$\vdots$$
$$= \lambda^p x$$

Because $x$ is an eigenvector, it is nonzero, and therefore Theorem SMEZV tells us that $\lambda^p = 0$ and so $\lambda = 0$. 
