Show all of your work and explain your answers fully. There is a total of 90 possible points. If you use a calculator or Mathematica on a problem be sure to write down both the input and output.

1. Find all solutions to the system of equations below, and express your answer as a solution set. (15 points)

   Augmented matrix:
   
   \[
   \begin{bmatrix}
   2 & 1 & 3 & 5 & -4 \\
   1 & 1 & 4 & 1 \\
   3 & 2 & 4 & 9 & -1
   \end{bmatrix}
   \]

   Row reduces to
   
   \[
   \begin{bmatrix}
   1 & 0 & 2 & 1 & 0 \\
   0 & 1 & -1 & 3 & 0 \\
   0 & 0 & 0 & 0 & 1
   \end{bmatrix}
   \]

   With a leading 1 in the final column we know that system is \( \text{inconsistent} \) (Thm RCLS)

   Solution set: \( \emptyset \)

2. Find all solutions to the system of equations below, and express your answer as a solution set. (15 points)

   Augmented matrix
   
   \[
   \begin{bmatrix}
   1 & -2 & 2 & -4 & 8 \\
   -1 & 2 & 1 & -5 & -5 \\
   1 & -2 & 0 & 2 & 6
   \end{bmatrix}
   \]

   Row reduce to
   
   \[
   \begin{bmatrix}
   1 & -2 & 0 & 2 & 6 \\
   0 & 0 & 1 & -3 & 1 \\
   0 & 0 & 0 & 0 & 0
   \end{bmatrix}
   \]

   \( D = 7134 \) \( \text{consistent} \)

   Solution set:
   
   \( (2, 4, 5) \) \( x_2, x_4 \) free
   
   \( \{ (6 + 2x_2 - 2x_4, x_2, 1 + 3x_4, x_4) | x_2, x_4 \in \mathbb{C} \} \)
3. Determine if the matrix below is singular or nonsingular. (15 points)

\[
A = \begin{bmatrix}
1 & -1 & 3 & 2 \\
1 & 3 & 2 & 0 \\
1 & 3 & 3 & -1 \\
2 & -2 & 7 & 3
\end{bmatrix}
\]

Row reduce the matrix and apply the NSRRT.

\[
\begin{bmatrix}
\text{ref} & 1 & 0 & 0 & 4.25 \\
0 & 1 & 0 & -0.75 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

This is not the 4x4 identity matrix, \( I_4 \)
(Definition IM)
so \( A \) is not nonsingular.

4. Convert the matrix below to reduced row-echelon form, doing all the computations by hand. In each step, indicate clearly which row operations you are performing. (15 points)

\[
\begin{bmatrix}
1 & 2 & -4 & -4 \\
1 & 1 & -3 & -3 \\
-2 & -1 & 6 & 8
\end{bmatrix}
\]

\(- R_1 + R_2 \rightarrow \begin{bmatrix}
1 & 2 & -4 & -4 \\
0 & 1 & -1 & -1 \\
-2 & -1 & 6 & 8
\end{bmatrix}
\]

\(2R_1 + R_3 \rightarrow \begin{bmatrix}
1 & 2 & -4 & -4 \\
0 & 1 & -1 & -1 \\
0 & 3 & -2 & 0
\end{bmatrix}
\]

\((3) R_2 \rightarrow \begin{bmatrix}
1 & 2 & -4 & -4 \\
0 & 1 & -1 & -1 \\
0 & 3 & -2 & 0
\end{bmatrix}
\]

\(-2R_2 + R_1 \rightarrow \begin{bmatrix}
1 & 0 & -2 & -2 \\
0 & 1 & -1 & -1 \\
0 & 3 & -2 & 0
\end{bmatrix}
\]

\(-3R_2 + R_3 \rightarrow \begin{bmatrix}
1 & 0 & -2 & -2 \\
0 & 1 & -1 & -1 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

\(R_3 + R_2 \rightarrow \begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]
5. A three-digit number has two properties. The tens-digit and the ones-digit add up to 5. If the number is written with the digits in the reverse order, and then subtracted from the original number, the result is 792. Use a system of equations to find all of the three-digit numbers with these properties. (15 points)

Solution: Let $a$ be the hundreds digit, $b$ the tens digit, and $c$ the ones digit. Then the first condition says that $b + c = 5$. The original number is $100a + 10b + c$, while the reversed number is $100c + 10b + a$. So the second condition is

$$792 = (100a + 10b + c) - (100c + 10b + a) = 99a - 99c$$

So we arrive at the system of equations

$$b + c = 5$$
$$99a - 99c = 792$$

Using row operations, or an augmented matrix, we arrive at the equivalent system

$$a - c = 8$$
$$b + c = 5$$

We can vary $c$ and obtain infinitely many solutions. However, $c$ must be a digit, restricting us to ten values (0 – 9). Furthermore, if $c > 2$, then the first equation causes $a > 9$, an impossibility. Setting $c = 0$, yields 850 as a solution, and setting $c = 1$ yields 941 as another solution.

6. Suppose that $A$ is a singular matrix, and $B$ is a matrix in reduced row-echelon form that is row-equivalent to $A$. Prove that the last row of $B$ is a zero row. (15 points)

Solution: Let $n$ denote the size of the square matrix $A$. By Theorem NSRRI the hypothesis that $A$ is singular implies that $B$ is not the identity matrix $I_n$. If $B$ has $n$ pivot columns, then it would have to be $I_n$, so $B$ must have fewer than $n$ pivot columns. But the number of nonzero rows in $B$ ($r$) is equal to the number of pivot columns as well. So the $n$ rows of $B$ have fewer than $n$ nonzero rows, and $b$ must contain at least one zero row. By Definition RREF, this row must be at the bottom of $B$. 