Math 232 Name: Key Dr. Beezer
Quiz R Spring 2005

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points.

1. Find a matrix representation of the linear transformation $T$ relative to the bases $B$ and $C$. (15 points)

$$T\colon P_2 \mapsto \mathbb{C}^2, \quad T(p(x)) = \begin{bmatrix} p(1) \\ p(3) \end{bmatrix}$$
$$B = \left\{2 - 5x + x^2,\ 1 + x - x^2,\ x^2\right\}$$
$$C = \left\{\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$$

Solution: Applying Definition MR,

$$\rho_C\left(T\left(2 - 5x + x^2\right)\right) = \rho_C\left(\begin{bmatrix} -2 \\ -4 \end{bmatrix}\right) = \rho_C\left(2\begin{bmatrix} 3 \\ 4 \end{bmatrix} + (-4)\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$
$$\rho_C\left(T\left(1 + x - x^2\right)\right) = \rho_C\left(\begin{bmatrix} 1 \\ -5 \end{bmatrix}\right) = \rho_C\left(13\begin{bmatrix} 3 \\ 4 \end{bmatrix} + (-19)\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 13 \\ -19 \end{bmatrix}$$
$$\rho_C\left(T\left(x^2\right)\right) = \rho_C\left(\begin{bmatrix} 1 \\ 9 \end{bmatrix}\right) = \rho_C\left((-15)\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 23\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -15 \\ 23 \end{bmatrix}$$

So the resulting matrix representation is

$$M^T_{B,C} = \begin{bmatrix} 2 & 13 & -15 \\ -4 & -19 & 23 \end{bmatrix}$$

2. Prove that the linear transformation $S$ is invertible. Then find a formula for the inverse linear transformation, $S^{-1}$, by employing a matrix inverse. (15 points)

$$S\colon P_1 \mapsto M_{1,2}, \quad S(a + bx) = \begin{bmatrix} 3a + b & 2a + b \end{bmatrix}$$

Solution: First, build a matrix representation of $S$ (Definition MR). We are free to choose whatever bases we wish, so we should choose ones that are easy to work with, such as

$$B = \{1, x\}$$
$$C = \left\{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}\right\}$$

The resulting matrix representation is then

$$M^T_{B,C} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

this matrix is invertible, since it has a nonzero determinant, so by Theorem XX the linear transformation $S$ is invertible. We can us the matrix inverse and Theorem IMR to find a formula for the inverse linear transformation,

$$\begin{aligned}
S^{-1}\left(\begin{bmatrix}a & b\end{bmatrix}\right) &= \rho_B^{-1}\left(M_{C,B}^{S^{-1}}\rho_C\left(\begin{bmatrix}a & b\end{bmatrix}\right)\right) && \text{Theorem FTMR}\\
&= \rho_B^{-1}\left(\left(M_{B,C}^{S}\right)^{-1}\rho_C\left(\begin{bmatrix}a & b\end{bmatrix}\right)\right) && \text{Theorem IMR}\\
&= \rho_B^{-1}\left(\left(M_{B,C}^{S}\right)^{-1}\begin{bmatrix}a\\ b\end{bmatrix}\right) && \text{Definition VR}\\
&= \rho_B^{-1}\left(\left(\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}a\\ b\end{bmatrix}\right) && \\
&= \rho_B^{-1}\left(\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}\right) && \text{Definition MI}\\
&= \rho_B^{-1}\left(\begin{bmatrix}a-b\\ -2a+3b\end{bmatrix}\right) && \text{Definition MVP}\\
&= (a-b) + (-2a+3b)x && \text{Definition VR}
\end{aligned}$$

3. Let $S_{22}$ be the vector space of $2\times 2$ symmetric matrices. Find a basis $B$ for $S_{22}$ that yields a diagonal matrix representation of the linear transformation $R$. (15 points)

$$R\colon S_{22}\mapsto S_{22},\quad R\left(\begin{bmatrix}a & b\\ b & c\end{bmatrix}\right) = \begin{bmatrix}-5a+2b-3c & -12a+5b-6c\\ -12a+5b-6c & 6a-2b+4c\end{bmatrix}$$

Solution: Begin with a matrix representation of $R$, any matrix representation, but use the same basis for both instances of $S_{22}$. We'll choose a basis that makes it easy to compute vector representations in $S_{22}$.

$$B = \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$$

Then the resulting matrix representation of $R$ (Theorem MR) is

$$M_{B,B}^{R} = \begin{bmatrix}-5 & 2 & -3\\ -12 & 5 & -6\\ 6 & -2 & 4\end{bmatrix}$$

Now, compute the eigenvalues and eigenvectors of this matrix, with the goal of diagonalizing the matrix (Theorem DC),

$$\lambda = 2 \qquad E_{M_{B,B}^{R}}(2) = \mathcal{S}p\left(\left\{\begin{bmatrix}-1\\ -2\\ 1\end{bmatrix}\right\}\right)$$

$$\lambda = 1 \qquad E_{M_{B,B}^{R}}(1) = \mathcal{S}p\left(\left\{\begin{bmatrix}-1\\ 0\\ 2\end{bmatrix},\ \begin{bmatrix}1\\ 3\\ 0\end{bmatrix}\right\}\right)$$

The three vectors that occur as basis elements for these eigenspaces will together form a linearly independent set (check this!). So these column vectors may be employed in a matrix that will diagonalize the matrix representation. If we "un-coordinatize" these three column vectors relative to the basis $B$, we will find three linearly independent elements of $S_{22}$ that are eigenvectors of the linear transformation $R$ (Theorem EER). A matrix representation relative to this basis of eigenvectors will be diagonal, with the eigenvalues ($\lambda = 2,\, 1$) as the diagonal elements. Here we go,

$$\rho_B^{-1}\left(\begin{bmatrix}-1\\-2\\1\end{bmatrix}\right) = (-1)\begin{bmatrix}1&0\\0&0\end{bmatrix} + (-2)\begin{bmatrix}0&1\\1&0\end{bmatrix} + 1\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}-1&-2\\-2&1\end{bmatrix}$$

$$\rho_B^{-1}\left(\begin{bmatrix}-1\\0\\2\end{bmatrix}\right) = (-1)\begin{bmatrix}1&0\\0&0\end{bmatrix} + 0\begin{bmatrix}0&1\\1&0\end{bmatrix} + 2\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}-1&0\\0&2\end{bmatrix}$$

$$\rho_B^{-1}\left(\begin{bmatrix}1\\3\\0\end{bmatrix}\right) = 1\begin{bmatrix}1&0\\0&0\end{bmatrix} + 3\begin{bmatrix}0&1\\1&0\end{bmatrix} + 0\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}1&3\\3&0\end{bmatrix}$$

So the requested basis of $S_{22}$, yielding a diagonal matrix representation of $R$, is

$$\left\{\begin{bmatrix}-1&-2\\-2&1\end{bmatrix}\ \begin{bmatrix}-1&0\\0&2\end{bmatrix},\ \begin{bmatrix}1&3\\3&0\end{bmatrix}\right\}$$

4. Use a matrix representation to find a basis for the range of the linear transformation $L$. (15 points)

$$L\colon M_{22} \mapsto P_2, \quad T\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = (a+2b+4c+d) + (3a+c-2d)x + (-a+b+3c+3d)x^2$$

Solution: As usual, build any matrix representation of $L$, most likely using a "nice" bases, such as

$$B = \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\ \begin{bmatrix}0&1\\0&0\end{bmatrix},\ \begin{bmatrix}0&0\\1&0\end{bmatrix},\ \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$

$$C = \left\{1,\, x,\, x^2\right\}$$

Then the matrix representation (Theorem MR) is,

$$M^L_{B,C} = \begin{bmatrix}1&2&4&1\\3&0&1&-2\\-1&1&3&3\end{bmatrix}$$

Theorem IR tells us that we can compute the range of the matrix representation, then use the isomorphism $\rho_C^{-1}$ to convert the range of the matrix representation into the range of the linear transformation. So we first analyze the matrix representation,

$$\begin{bmatrix}1&2&4&1\\3&0&1&-2\\-1&1&3&3\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1}&0&0&-1\\0&\boxed{1}&0&-1\\0&0&\boxed{1}&1\end{bmatrix}$$

With three nonzero rows in the reduced row-echelon form of the matrix, we know the range has dimension 3. Since $P_2$ has dimension 3 (Theorem DP), the range must be all of $P_2$. So *any* basis of $P_2$ would suffice as a basis for the range. For instance, $C$ itself would be a correct answer.

A more laborious approach would be to use Theorem BROC and choose the first three columns of the matrix representation as a basis for the range of the matrix representation. These could then be "un-coordinatized" with $\rho_C^{-1}$ to yield a ("not nice") basis for $P_2$.

5. The linear transformation $D$ performs differentiation on polynomials. Use a matrix representation of $D$ to find the rank and nullity of $D$. (15 points)

$$D\colon P_n \mapsto P_n, \quad D\left(p(x)\right) = p'(x)$$

Solution: Build a matrix representation (Theorem MR) with the set

$$B = \left\{1,\, x,\, x^2,\, \ldots,\, x^n\right\}$$

employed as a basis of both the domain and codomain. Then

$$\rho_B\left(D\left(1\right)\right) = \rho_B\left(0\right) = \begin{bmatrix}0\\0\\0\\\vdots\\0\\0\end{bmatrix} \qquad \rho_B\left(D\left(x\right)\right) = \rho_B\left(1\right) = \begin{bmatrix}1\\0\\0\\\vdots\\0\\0\end{bmatrix}$$

$$\rho_B\left(D\left(x^2\right)\right) = \rho_B\left(2x\right) = \begin{bmatrix}0\\2\\0\\\vdots\\0\\0\end{bmatrix} \qquad \rho_B\left(D\left(x^3\right)\right) = \rho_B\left(3x^2\right) = \begin{bmatrix}0\\0\\3\\\vdots\\0\\0\end{bmatrix}$$

$$\vdots$$

$$\rho_B\left(D\left(x^n\right)\right) = \rho_B\left(nx^{n-1}\right) = \begin{bmatrix}0\\0\\0\\\vdots\\n\\0\end{bmatrix}$$

and the resulting matrix representation is

$$M^D_{B,B} = \begin{bmatrix}0 & 1 & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 2 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & 3 & \dots & 0 & 0\\ & \vdots & & & \ddots & & \vdots\\ 0 & 0 & 0 & 0 & \dots & 0 & n\\ 0 & 0 & 0 & 0 & \dots & 0 & 0\end{bmatrix}$$

This $(n+1)\times(n+1)$ matrix is very close to being in reduced row-echelon form. Multiply row $i$ by $\frac{1}{i}$, for $1 \leq i \leq n$, to convert it to reduced row-echelon form. From this we can see that matrix representation $M^D_{B,B}$ has rank $n$ and nullity 1. Applying Theorem INS and Theorem IR tells us that the linear transformation $D$ will have the same values for the rank and nullity, as well.

6. Suppose that $B$ is a basis of the vector space $V$. Prove that vector representation, $\rho_B$, is injective. (You may assume that $\rho_B$ is a linear transformation.) (15 points)

Solution: This is Theorem VRI. See the proof given there.