Math 232 Name: Key Dr. Beezer
Quiz LT Spring 2005

Show *all* of your work and *explain* your answers fully. There is a total of 105 possible points.

1. Verify that the function below is a linear transformation. (15 points)

$$T\colon P_2 \mapsto \mathbb{C}^2, \quad T\left(a + bx + cx^2\right) = \begin{bmatrix} 2a - b \\ b + c \end{bmatrix}$$

Solution: Check the two conditions of Definition LT.

$$\begin{aligned}
T(\mathbf{u} + \mathbf{v}) &= T\left(\left(a + bx + cx^2\right) + \left(d + ex + fx^2\right)\right) \\
&= T\left((a + d) + (b + e)\,x + (c + f)\,x^2\right) \\
&= \begin{bmatrix} 2(a + d) - (b + e) \\ (b + e) + (c + f) \end{bmatrix} \\
&= \begin{bmatrix} (2a - b) + (2d - e) \\ (b + c) + (e + f) \end{bmatrix} \\
&= \begin{bmatrix} 2a - b \\ b + c \end{bmatrix} + \begin{bmatrix} 2d - e \\ e + f \end{bmatrix} \\
&= T(\mathbf{u}) + T(\mathbf{v})
\end{aligned}$$

and

$$\begin{aligned}
T(\alpha\mathbf{u}) &= T\left(\alpha\left(a + bx + cx^2\right)\right) \\
&= T\left((\alpha a) + (\alpha b)\,x + (\alpha c)\,x^2\right) \\
&= \begin{bmatrix} 2(\alpha a) - (\alpha b) \\ (\alpha b) + (\alpha c) \end{bmatrix} \\
&= \begin{bmatrix} \alpha(2a - b) \\ \alpha(b + c) \end{bmatrix} \\
&= \alpha \begin{bmatrix} (2a - b) \\ (b + c)) \end{bmatrix} \\
&= \alpha T(\mathbf{u})
\end{aligned}$$

So T is indeed a linear transformation.

2. Consider the linear transformation S, and compute the following pre-images. (15 points)

$$S\colon \mathbb{C}^3 \mapsto \mathbb{C}^3, \quad S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a - 2b - c \\ 3a - b + 2c \\ a + b + 2c \end{bmatrix}$$

(a) $S^{-1}\left(\begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}\right)$

Solution: We work from the definition of the pre-image, Definition PI. Setting

$$S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$$

we arrive at a system of three equations in three variables, with an augmented matrix that we row-reduce in a search for solutions,

$$\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -1 & 2 & 5 \\ 1 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

With a leading 1 in the last column, this system is inconsistent (Theorem RCLS), and there are no values of a, b and c that will create an element of the pre-image. So the preimage is the empty set.

(b) $S^{-1}\left(\begin{bmatrix} -5 \\ 5 \\ 7 \end{bmatrix}\right)$

Solution: We work from the definition of the pre-image, Definition PI. Setting

$$S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 5 \\ 7 \end{bmatrix}$$

we arrive at a system of three equations in three variables, with an augmented matrix that we row-reduce in a search for solutions,

$$\begin{bmatrix} 1 & -2 & -1 & -5 \\ 3 & -1 & 2 & 5 \\ 1 & 1 & 2 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution set to this system, which is also the desired pre-image, can be expressed using the vector form of the solutions (Theorem VFSLS)

$$S^{-1}\left(\begin{bmatrix} -5 \\ 5 \\ 7 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \middle|\, c \in \mathbb{C} \right\} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \mathcal{S}p\left(\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}\right)$$

Does the final expression for this set remind you of Theorem NSPI?

3. Show that the linear transformation R is not injective by finding two different elements of the domain, $\mathbf{x}$ and $\mathbf{y}$, such that $R(\mathbf{x}) = R(\mathbf{y})$. ($S_{22}$ is the vector space of symmetric 2×2 matrices.) (15 points)

$$R\colon S_{22} \mapsto P_1 \quad R\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = (2a - b + c) + (a + b + 2c)x$$

Solution: We choose $\mathbf{x}$ to be any vector we like. A particularly cocky choice would be to choose $\mathbf{x} = \mathbf{0}$, but we will instead choose

$$\mathbf{x} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$$

Then $R(\mathbf{x}) = 9 + 9x$. Now compute the null space of R, which by Theorem NSILT we expect to be nontrivial. Setting $R\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right)$ equal to the zero vector, $\mathbf{0} = 0 + 0x$, and equating coefficients leads to a homogenous system of equations. Row-reducing the coefficient matrix of this system will allow us to determine the values of a, b and c that create elements of the null space of R,

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \end{bmatrix}$$

We only need a single element of the null space, so we will not compute a precise description of the whole null space. Instead, choose the free variable $c = 2$. Then

$$\mathbf{z} = \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}$$

is the corresponding element of the null space. We compute the desired $\mathbf{y}$ as

$$\mathbf{y} = \mathbf{x} + \mathbf{y} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -3 & 6 \end{bmatrix}$$

Then check that $R(\mathbf{y}) = 9 + 9x$.

4. Show that the linear transformation T is not surjective by finding an element of the codomain, $\mathbf{v}$, such that there is no vector $\mathbf{u}$ with $T(\mathbf{u}) = \mathbf{v}$. (15 points)

$$T\colon \mathbb{C}^3 \mapsto \mathbb{C}^3, \quad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2a + 3b - c \\ 2b - 2c \\ a - b + 2c \end{bmatrix}$$

Solution: We wish to find an output vector $\mathbf{v}$ that has no associated input. This is the same as requiring that there is no solution to the equality

$$\mathbf{v} = T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2a + 3b - c \\ 2b - 2c \\ a - b + 2c \end{bmatrix} = a\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + c\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

In other words, we would like to find an element of $\mathbb{C}^3$ not in the set

$$Y = \mathcal{S}p\left(\left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}\right\}\right)$$

If we make these vectors the rows of a matrix, and row-reduce, Theorem BRS provides an alternate description of Y,

$$Y = \mathcal{S}p\left(\left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -5 \end{bmatrix}\right\}\right)$$

If we add these vectors together, and then change the third component of the result, we will create a vector that lies outside of Y, say $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 9 \end{bmatrix}$.

5. The linear transformation S is invertible. Find a formula for the inverse linear transformation, S^{-1}. (15 points)

$$S\colon P_1 \mapsto M_{1,2}, \quad S(a + bx) = \begin{bmatrix} 3a + b & 2a + b \end{bmatrix}$$

Solution: Suppose that $S^{-1}\colon M_{1,2} \mapsto P_1$ has a form given by

$$S^{-1}\begin{pmatrix} z & w \end{pmatrix} = (rz + sw) + (pz + qw)x$$

where r, s, p, q are unknown scalars. Then

$$\begin{aligned} a + bx &= S^{-1}\left(S\left(a + bx\right)\right) \\ &= S^{-1}\left(\begin{bmatrix} 3a + b & 2a + b \end{bmatrix}\right) \\ &= \left(r(3a + b) + s(2a + b)\right) + \left(p(3a + b) + q(2a + b)\right) x \\ &= \left((3r + 2s)a + (r + s)b\right) + \left((3p + 2q)a + (p + q)b\right) x \end{aligned}$$

Equating coefficients of these two polynomials, and then equating coefficients on a and b, gives rise to 4 equations in 4 variables,

$$\begin{aligned} 3r + 2s &= 1 \\ r + s &= 0 \\ 3p + 2q &= 0 \\ p + q &= 1 \end{aligned}$$

This system has a unique solution: $r = 1$, $s = -1$, $p = -2$, $q = 3$. So the desired inverse linear transformation is

$$S^{-1}\left(\begin{matrix} z & w \end{matrix}\right) = (z - w) + (-2z + 3w)\, x$$

Notice that the system of 4 equations in 4 variables could be split into two systems, each with two equations in two variables (and identical coefficient matrices). After making this split, the solution might feel like computing the inverse of a matrix (Theorem CINSM).

6. Suppose that A is an $m \times n$ matrix. Define the linear transformation T by

$$T\colon \mathbb{C}^n \mapsto \mathbb{C}^m, \quad T\left(\mathbf{x}\right) = A\mathbf{x}$$

Prove that the null space of T equals the null space of A, $\mathcal{N}(A) = \mathcal{N}(T)$. (15 points)

Solution: This is an equality of sets, so we want to establish two subset conditions (Technique SE).

First, show $\mathcal{N}(A) \subseteq \mathcal{N}(T)$. Choose $\mathbf{x} \in \mathcal{N}(A)$. Check to see if $\mathbf{x} \in \mathcal{N}(T)$,

$$\begin{aligned} T\left(\mathbf{x}\right) &= A\mathbf{x} && \text{Definition of } T \\ &= \mathbf{0} && \mathbf{x} \in \mathcal{N}(A) \end{aligned}$$

So by Definition NSLT, $\mathbf{x} \in \mathcal{N}(T)$ and $\mathcal{N}(A) \subseteq \mathcal{N}(T)$.

Now, show $\mathcal{N}(T) \subseteq \mathcal{N}(A)$. Choose $\mathbf{x} \in \mathcal{N}(T)$. Check to see if $\mathbf{x} \in \mathcal{N}(A)$,

$$\begin{aligned} A\mathbf{x} &= T\left(\mathbf{x}\right) && \text{Definition of } T \\ &= \mathbf{0} && \mathbf{x} \in \mathcal{N}(T) \end{aligned}$$

So by Definition NSM, $\mathbf{x} \in \mathcal{N}(A)$ and $\mathcal{N}(T) \subseteq \mathcal{N}(A)$.

7. Suppose that that $T\colon U \mapsto V$ and $S\colon V \mapsto W$ are linear transformations. Prove the following relationship between ranges. (15 points)

$$\mathcal{R}(S \circ T) \subseteq \mathcal{R}(S)$$

Solution: This question asks us to establish that one set ($\mathcal{R}(S \circ T)$) is a subset of another ($\mathcal{R}(S)$). Choose an element in the "smaller" set, say $\mathbf{w} \in \mathcal{R}(S \circ T)$. Then we know that there is a vector $\mathbf{u} \in U$ such that

$$\mathbf{w} = \left(S \circ T\right)\left(\mathbf{u}\right) = S\left(T\left(\mathbf{u}\right)\right)$$

Now define $\mathbf{v} = T(\mathbf{u})$, so that then

$$S(\mathbf{v}) = S(T(\mathbf{u})) = \mathbf{w}$$

This statement is sufficient to show that $\mathbf{w} \in \mathcal{R}(S)$, so $\mathbf{w}$ is an element of the "larger" set, and $\mathcal{R}(S \circ T) \subseteq \mathcal{R}(S)$.