Show all of your work and explain your answers fully. There is a total of 90 possible points.

1. Consider the subspace
   \[ W = \text{span}\left(\left\{ \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \right\} \right) \]
   of the vector space of 2 \times 2 matrices, \( M_{22} \). Is \( C = \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix} \) an element of \( W \)? (15 points)

   Solution: In order to belong to \( W \), we must be able to express \( C \) as a linear combination of the elements in the spanning set of \( W \). So we begin with such an expression, using the unknowns \( a, b, c \) for the scalars in the linear combination.
   \[
   C = \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix} = a \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} + b \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix} + c \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}
   \]

   Massaging the right-hand side, according to the definition of the vector space operations in \( M_{22} \) (Example VSM), we find the matrix equality,
   \[
   \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 2a + 4b - 3c & a + c \\ 3a + 2b + 2c & -a + 3b + c \end{bmatrix}
   \]

   Matrix equality allows us to form a system of four equations in three variables, whose augmented matrix row-reduces as follows,
   \[
   \begin{bmatrix} 2 & 4 & -3 & -3 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & 2 & 6 \\ -1 & 3 & 1 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
   \]

   Since this system of equations is consistent (Theorem RCLS), a solution will provide values for \( a, b \) and \( c \) that allow us to recognize \( C \) as an element of \( W \).

2. Determine if the set \( T = \{x^2 - x + 5, 4x^3 - x^2 + 5x, 3x + 2\} \) spans the vector space of polynomials with degree 4 or less, \( P_4 \). (15 points)

   Solution: The vector space \( P_4 \) has dimension 5 by Theorem DP. Since \( T \) contains only 3 vectors, and 3 < 5, Theorem G tells us that \( T \) does not span \( P_5 \).
3. In the crazy vector space $C$ (Example CVS), is the set $S = \{(0, 2), (2, 8)\}$ linearly independent? (15 points)

Solution: We begin with a relation of linear dependence using unknown scalars $a$ and $b$. We wish to know if these scalars must both be zero. Recall that the zero vector in $C$ is $(-1, -1)$ and that the definitions of vector addition and scalar multiplication are not what we might expect.

$$0 = (-1, -1) = a(0, 2) + b(2, 8)$$

$$= (0a + a - 1, 2a + a - 1) + (2b + b - 1, 8b + b - 1)$$

$$= (a - 1, 3a - 1) + (3b - 1, 9b - 1)$$

$$= (a - 1 + 3b - 1 + 1, 3a - 1 + 9b - 1 + 1)$$

$$= (a + 3b - 1, 3a + 9b - 1)$$

From this we obtain two equalities, which can be converted to a homogeneous system of equations,

$$-1 = a + 3b - 1$$

$$-1 = 3a + 9b - 1$$

This homogeneous system has a singular coefficient matrix (Theorem SMZD), and so has more than just the trivial solution (Definition NM). Any nontrivial solution will give us a nontrivial relation of linear dependence on $S$. So $S$ is linearly dependent (Definition LI).

4. A $2 \times 2$ matrix $B$ is upper-triangular if $[B]_{21} = 0$. Let $UT_2$ be the set of all $2 \times 2$ upper-triangular matrices. Then $UT_2$ is a subspace of the vector space of all $2 \times 2$ matrices, $M_{22}$ (you may assume this). Determine the dimension of $UT_2$ providing all of the necessary justifications for your answer. (15 points)

Solution: A typical matrix from $UT_2$ looks like

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where $a, b, c \in \mathbb{C}$ are arbitrary scalars. Observing this we can then write

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which says that

$$R = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a spanning set for $UT_2$ (Definition TSS). Is $R$ is linearly independent? If so, it is a basis for $UT_2$. So consider a relation of linear dependence on $R$,

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

From this equation, one rapidly arrives at the conclusion that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. So $R$ is a linearly independent set (Definition LI), and hence is a basis (Definition B) for $UT_2$. Now, we simply count up the size of the set $R$ to see that the dimension of $UT_2$ is $\dim (UT_2) = 3$. 


5. A square matrix \( A \) of size \( n \) is upper-triangular if \( [A]_{ij} = 0 \) whenever \( i > j \). Let \( UT_n \) be the set of all upper-triangular matrices of size \( n \). Prove that \( UT_n \) is a subspace of the vector space of all square matrices of size \( n \), \( M_{nn} \). (15 points)

Solution: Apply Theorem TSS.

First, the zero vector of \( M_{nn} \) is the zero matrix, \( \mathcal{O} \), whose entries are all zero (Definition ZM). This matrix then meets the condition that \( [\mathcal{O}]_{ij} = 0 \) for \( i > j \) and so is an element of \( UT_n \).

Suppose \( A, B \in UT_n \). Is \( A + B \in UT_n \)? We examine the entries of \( A + B \) “below” the diagonal. That is, in the following, assume that \( i > j \).

\[
\begin{align*}
[A + B]_{ij} &= [A]_{ij} + [B]_{ij} \\
&= 0 + 0 \\
&= 0
\end{align*}
\]  

Definition MA  
\( A, B \in UT_n \)

which qualifies \( A + B \) for membership in \( UT_n \).

Suppose \( \alpha \in \mathbb{C} \) and \( A \in UT_n \). Is \( \alpha A \in UT_n \)? We examine the entries of \( \alpha A \) “below” the diagonal. That is, in the following, assume that \( i > j \).

\[
\begin{align*}
[\alpha A]_{ij} &= \alpha [A]_{ij} \\
&= \alpha 0 \\
&= 0
\end{align*}
\]  

Definition MSM  
\( A \in UT_n \)

which qualifies \( \alpha A \) for membership in \( UT_n \).

Having fulfilled the three conditions of Theorem TSS we see that \( UT_n \) is a subspace of \( M_{nn} \).

6. Suppose that \( V \) is a vector space. Then by Property AI we know that for every vector \( \mathbf{v} \in V \), there is an additive inverse \( -\mathbf{v} \in V \). Prove that the additive inverse is unique for each choice of \( \mathbf{v} \). (15 points)

Solution: This is Theorem AIU. A careful proof can be found there.