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Quiz R Fall 2005

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points. If you use a calculator on a problem be sure to write down both the input to, and output from, the calculator.

1. Let S_{22} be the vector space of 2×2 symmetric matrices. Build the matrix representation of the linear transformation $T\colon P_2 \mapsto S_{22}$ relative to the bases B and C and then use this matrix representation to compute $T\left(3+5x-2x^2\right)$. (15 points)

$$B=\left\{1,\,1+x,\,1+x+x^2\right\} \qquad C=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$

$$T\left(a+bx+cx^2\right)=\begin{bmatrix}2a-b+c & a+3b-c\\ a+3b-c & a-c\end{bmatrix}$$

Solution: Input to T the vectors of the basis B and coordinatize the outputs relative to C,

$$\rho_C\left(T\left(1\right)\right)=\rho_C\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\right)=\rho_C\left(2\begin{bmatrix}1&0\\0&0\end{bmatrix}+1\begin{bmatrix}0&1\\1&0\end{bmatrix}+1\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$\rho_C\left(T\left(1+x\right)\right)=\rho_C\left(\begin{bmatrix}1&4\\4&1\end{bmatrix}\right)=\rho_C\left(1\begin{bmatrix}1&0\\0&0\end{bmatrix}+4\begin{bmatrix}0&1\\1&0\end{bmatrix}+1\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}1\\4\\1\end{bmatrix}$$

$$\rho_C\left(T\left(1+x+x^2\right)\right)=\rho_C\left(\begin{bmatrix}2&3\\3&0\end{bmatrix}\right)=\rho_C\left(2\begin{bmatrix}1&0\\0&0\end{bmatrix}+3\begin{bmatrix}0&1\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}2\\3\\0\end{bmatrix}$$

Applying Definition MR we have the matrix representation

$$M^T_{B,C}=\begin{bmatrix}2&1&2\\1&4&3\\1&1&0\end{bmatrix}$$

To compute $T\left(3+5x-2x^2\right)$ employ Theorem FTMR,

$$\begin{aligned}T\left(3+5x-2x^2\right)&=\rho_C^{-1}\left(M^T_{B,C}\rho_B\left(3+5x-2x^2\right)\right)\\
&=\rho_C^{-1}\left(M^T_{B,C}\rho_B\left((-2)(1)+7(1+x)+(-2)(1+x+x^2)\right)\right)\\
&=\rho_C^{-1}\left(\begin{bmatrix}2&1&2\\1&4&3\\1&1&0\end{bmatrix}\begin{bmatrix}-2\\7\\-2\end{bmatrix}\right)\\
&=\rho_C^{-1}\left(\begin{bmatrix}-1\\20\\5\end{bmatrix}\right)\\
&=(-1)\begin{bmatrix}1&0\\0&0\end{bmatrix}+20\begin{bmatrix}0&1\\1&0\end{bmatrix}+5\begin{bmatrix}0&0\\0&1\end{bmatrix}\\
&=\begin{bmatrix}-1&20\\20&5\end{bmatrix}\end{aligned}$$

You can, of course, check your answer by evaluating $T\left(3+5x-2x^2\right)$ directly.

2. Use a matrix representation to determine if the linear transformation $T\colon P_3 \mapsto M_{22}$ surjective. (15 points)

$$T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} -a + 4b + c + 2d & 4a - b + 6c - d \\ a + 5b - 2c + 2d & a + 2c + 5d \end{bmatrix}$$

Solution: Choose bases B and C for the matrix representation,

$$B = \left\{1,\, x,\, x^2,\, x^3\right\} \qquad\qquad C = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Input to T the vectors of the basis B and coordinatize the outputs relative to C,

$$\rho_C\left(T\left(1\right)\right) = \rho_C\left(\begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}\right) = \rho_C\left((-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$$\rho_C\left(T\left(x\right)\right) = \rho_C\left(\begin{bmatrix} 4 & -1 \\ 5 & 0 \end{bmatrix}\right) = \rho_C\left(4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \\ 5 \\ 0 \end{bmatrix}$$

$$\rho_C\left(T\left(x^2\right)\right) = \rho_C\left(\begin{bmatrix} 1 & 6 \\ -2 & 2 \end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 6 \\ -2 \\ 2 \end{bmatrix}$$

$$\rho_C\left(T\left(x^3\right)\right) = \rho_C\left(\begin{bmatrix} 2 & -1 \\ 2 & 5 \end{bmatrix}\right) = \rho_C\left(2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 5 \end{bmatrix}$$

Applying Definition MR we have the matrix representation

$$M^T_{B,C} = \begin{bmatrix} -1 & 4 & 1 & 2 \\ 4 & -1 & 6 & -1 \\ 1 & 5 & -2 & 2 \\ 1 & 0 & 2 & 5 \end{bmatrix}$$

Properties of this matrix representation will translate to properties of the linear transformation The matrix representation is nonsingular since it row-reduces to the identity matrix (Theorem NSRRI) and therefore has a column space equal to $\mathbb{C}^4$ (Theorem CNSMB). The column space of the matrix representation is isomorphic to the range of the linear transformation (Theorem RCSI). So the range of T has dimension 4, equal to the dimension of the codomain M_{22}. By Theorem ROSLT, T is surjective.

3. Find a basis for the kernel of the linear transformation $T\colon P_2 \mapsto M_{22}$.

$$T\left(a+bx+cx^2\right) = \begin{bmatrix} a+2b-2c & 2a+2b \\ -a+b-4c & 3a+2b+2c \end{bmatrix}$$

Solution: Choose bases B and C for the matrix representation,

$$B = \left\{1,\, x,\, x^2\right\} \qquad C = \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$

Input to T the vectors of the basis B and coordinatize the outputs relative to C,

$$\rho_C\left(T\left(1\right)\right) = \rho_C\left(\begin{bmatrix}1&2\\-1&3\end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix}1&0\\0&0\end{bmatrix} + 2\begin{bmatrix}0&1\\0&0\end{bmatrix} + (-1)\begin{bmatrix}0&0\\1&0\end{bmatrix} + 3\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\-1\\3\end{bmatrix}$$

$$\rho_C\left(T\left(x\right)\right) = \rho_C\left(\begin{bmatrix}2&2\\1&2\end{bmatrix}\right) = \rho_C\left(2\begin{bmatrix}1&0\\0&0\end{bmatrix} + 2\begin{bmatrix}0&1\\0&0\end{bmatrix} + 1\begin{bmatrix}0&0\\1&0\end{bmatrix} + 2\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) = \begin{bmatrix}2\\2\\1\\2\end{bmatrix}$$

$$\rho_C\left(T\left(x^2\right)\right) = \rho_C\left(\begin{bmatrix}-2&0\\-4&2\end{bmatrix}\right) = \rho_C\left((-2)\begin{bmatrix}1&0\\0&0\end{bmatrix} + 0\begin{bmatrix}0&1\\0&0\end{bmatrix} + (-4)\begin{bmatrix}0&0\\1&0\end{bmatrix} + 2\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) = \begin{bmatrix}-2\\0\\-4\\2\end{bmatrix}$$

Applying Definition MR we have the matrix representation

$$M^T_{B,C} = \begin{bmatrix}1&2&-2\\2&2&0\\-1&1&-4\\3&2&2\end{bmatrix}$$

The null space of the matrix representation is isomorphic (via ρ_B) to the kernel of the linear transformation (Theorem KNSI). So we compute the null space of the matrix representation by first row-reducing the matrix to,

$$\begin{bmatrix}\boxed{1}&0&2\\0&\boxed{1}&-2\\0&0&0\\0&0&0\end{bmatrix}$$

Employing Theorem BNS we have

$$\mathcal{N}\left(M^T_{B,C}\right) = \mathcal{S}p\left(\left\{\begin{bmatrix}-2\\2\\1\end{bmatrix}\right\}\right)$$

We only need to uncoordinatize this one basis vector to get a basis for $\mathcal{K}(T)$,

$$\mathcal{K}(T) = \mathcal{S}p\left(\left\{\rho_B^{-1}\left(\begin{bmatrix}-2\\2\\1\end{bmatrix}\right)\right\}\right) = \mathcal{S}p\left(\left\{-2+2x+x^2\right\}\right)$$

4. The linear transformation $R\colon M_{12} \mapsto M_{21}$ is invertible. Use a matrix representation to determine a formula for the inverse linear transformation $R^{-1}\colon M_{21} \mapsto M_{12}$. (15 points)

$$R\left(\begin{bmatrix} a & b \end{bmatrix}\right) = \begin{bmatrix} a+3b \\ 4a+11b \end{bmatrix}$$

Solution: Choose bases B and C for M_{12} and M_{21} (respectively),

$$B = \left\{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}\right\} C = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

The resulting matrix representation is

$$M^{R}_{B,C} = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}$$

This matrix is invertible (its determinant is nonzero, Theorem SMZD), so by Theorem IMR, we can compute the matrix representation of R^{-1} with a matrix inverse (Theorem TTMI),

$$M^{R^{-1}}_{C,B} = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}^{-1} = \begin{bmatrix} -11 & 3 \\ 4 & -1 \end{bmatrix}$$

To obtain a general formula for R^{-1}, use Theorem FTMR,

$$R^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \rho_B\left(M^{R^{-1}}_{C,B}\,\rho_C\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = \rho_B\left(\begin{bmatrix} -11 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) = \rho_B\left(\begin{bmatrix} -11x+3y \\ 4x-y \end{bmatrix}\right) = \begin{bmatrix} -11x+3y & 4x-y \end{bmatrix}$$

5. Let S_{22} be the vector space of 2×2 symmetric matrices. Find a basis for S_{22} composed of eigenvectors of the linear transformation $Q\colon S_{22} \mapsto S_{22}$. (15 points)

$$Q\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = \begin{bmatrix} 25a+18b+30c & -16a-11b-20c \\ -16a-11b-20c & -11a-9b-12c \end{bmatrix}$$

Solution: Use a single basis for both the domain and codomain, since they are equal.

$$B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

The matrix representation of Q relative to B is

$$M = M^{Q}_{B,B} = \begin{bmatrix} 25 & 18 & 30 \\ -16 & -11 & -20 \\ -11 & -9 & -12 \end{bmatrix}$$

We can analyze this matrix with the techniques of Section EE and then apply Theorem EER. The eigenvalues of this matrix are $\lambda = -2, 1, 3$ with eigenspaces

$$\mathcal{E}_M(-2) = \mathcal{S}p\left(\left\{\begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}\right\}\right) \qquad \mathcal{E}_M(1) = \mathcal{S}p\left(\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}\right) \qquad \mathcal{E}_M(3) = \mathcal{S}p\left(\left\{\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}\right\}\right)$$

Because the three eigenvalues are distinct, the three basis vectors from the three eigenspaces for a linearly independent set (Theorem EDELI). Theorem EER says we can uncoordinatize these eigenvectors to obtain eigenvectors of Q. By Theorem ILTLI the resulting set will remain linearly independent. Set

$$C = \left\{\rho_B^{-1}\left(\begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}\right), \rho_B^{-1}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right), \rho_B^{-1}\left(\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}\right)\right\} = \left\{\begin{bmatrix} -6 & 4 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 2 \\ 2 & 1 \end{bmatrix}\right\}$$

Then C is a linearly independent set of size 3 in the vector space M_{22}, which has dimension 3 as well. By Theorem G, C is a basis of M_{22}.

6. Suppose that $T\colon U \mapsto V$ is a linear transformation and B and C are bases for U and V (respectively). The proof of Theorem KNSI shows that the vector spaces $\mathcal{K}(T)$ and $\mathcal{N}\left(M^T_{B,C}\right)$ are isomorphic by employing the isomorphism ρ_B. Show that ρ_B produces outputs in $\mathcal{N}\left(M^T_{B,C}\right)$ when supplied inputs from $\mathcal{K}(T)$. That is, prove that if $\mathbf{u} \in \mathcal{K}(T)$, then $\rho_B\left(\mathbf{u}\right) \in \mathcal{N}\left(M^T_{B,C}\right)$. (15 points)

Solution: Suppose that $\mathbf{u} \in \mathcal{K}(T)$. Then

$$\begin{aligned}
M^T_{B,C}\rho_B\left(\mathbf{u}\right) &= \rho_C\left(T\left(\mathbf{u}\right)\right) && \text{Theorem FTMR}\\
&= \rho_C\left(\mathbf{0}\right) && \mathbf{u} \in \mathcal{K}(T)\\
&= \mathbf{0} && \text{Theorem LTTZZ}
\end{aligned}$$

This says that $\rho_B\left(\mathbf{u}\right) \in \mathcal{N}\left(M^T_{B,C}\right)$, as desired.