Math 232 | Name: Key | Dr. Beezer
Quiz SLE | Code: | Fall 2005

Show *all* of your work and *explain* your answers fully. There is a total of 105 possible points. If you use a calculator on a problem be sure to write down both the input to, and output from, the calculator.

1. Solve the following system of equations, expressing the solution set with the correct notation. (15 points)

$$\begin{aligned} 2x_1 + 3x_2 - x_3 - 9x_4 &= -16 \\ x_1 + 2x_2 + x_3 &= 0 \\ -x_1 + 2x_2 + 3x_3 + 4x_4 &= 8 \end{aligned}$$

Solution: The augmented matrix of the system of equations is

$$\begin{bmatrix} 2 & 3 & -1 & -9 & -16 \\ 1 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 4 & 8 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 2 & 3 \\ 0 & \boxed{1} & 0 & -3 & -5 \\ 0 & 0 & \boxed{1} & 4 & 7 \end{bmatrix}$$

Then $D = \{1, 2, 3\}$ and $F = \{4, 5\}$, so the system is consistent ($5 \notin D$) and can be described by the one free variable x_4. Rearranging the equations represented by the three nonzero rows to gain expressions for the dependent variables x_1, x_2 and x_3, yields the solution set,

$$S = \left\{ \left. \begin{bmatrix} 3 - 2x_4 \\ -5 + 3x_4 \\ 7 - 4x_4 \\ x_4 \end{bmatrix} \right| x_4 \in \mathbb{C} \right\}$$

2. Solve the following system of equations, expressing the solution set with the correct notation. (15 points)

$$\begin{aligned} 2x_1 + 3x_2 + 19x_3 - 4x_4 &= 2 \\ x_1 + 2x_2 + 12x_3 - 3x_4 &= 1 \\ -x_1 + 2x_2 + 8x_3 - 5x_4 &= 1 \end{aligned}$$

Solution: The augmented matrix of the system of equations is

$$\begin{bmatrix} 2 & 3 & 19 & -4 & 2 \\ 1 & 2 & 12 & -3 & 1 \\ -1 & 2 & 8 & -5 & 1 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 1 & 0 \\ 0 & \boxed{1} & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

With a leading one in the last column Theorem RCLS tells us the system of equations is inconsistent, so the solution set is the empty set, $\emptyset = \{\}$.

3. Convert the matrix D to reduced row-echelon form by performing row operations *without* the aid of a calculator. Indicate clearly which row operations you are doing at each step. (15 points)

$$D = \begin{bmatrix} 1 & 2 & -4 \\ -3 & -1 & -3 \\ -2 & 1 & -7 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 2 & -4 \\ -3 & -1 & -3 \\ -2 & 1 & -7 \end{bmatrix} \xrightarrow{2R_1+R_2} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 5 & -15 \\ -2 & 1 & -7 \end{bmatrix} \xrightarrow{2R_1+R_3} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 5 & -15 \\ 0 & 5 & -15 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2}$$

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 5 & -15 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 5 & -15 \end{bmatrix} \xrightarrow{-5R_2+R_3} \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Compute the null space of the matrix A, $\mathcal{N}(A)$. (15 points)

$$A = \begin{bmatrix} 2 & 4 & 1 & 3 & 8 \\ -1 & -2 & -1 & -1 & 1 \\ 2 & 4 & 0 & -3 & 4 \\ 2 & 4 & -1 & -7 & 4 \end{bmatrix}$$

Solution: Definition NSM tells us that the null space of A is the solution set to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. The augmented matrix of this system is

$$\begin{bmatrix} 2 & 4 & 1 & 3 & 8 & 0 \\ -1 & -2 & -1 & -1 & 1 & 0 \\ 2 & 4 & 0 & -3 & 4 & 0 \\ 2 & 4 & -1 & -7 & 4 & 0 \end{bmatrix}$$

To solve the system, we row-reduce the augmented matrix and obtain,

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 5 & 0 \\ 0 & 0 & \boxed{1} & 0 & -8 & 0 \\ 0 & 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix represents a system with equations having three dependent variables (x_1, x_3, and x_4) and two independent variables (x_2 and x_5). These equations rearrange to

$$x_1 = -2x_2 - 5x_5 \qquad x_3 = 8x_5 \qquad x_4 = -2x_5$$

So we can write the solution set (which is the requested null space) as

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -2x_2 - 5x_5 \\ x_2 \\ 8x_5 \\ -2x_5 \\ x_5 \end{bmatrix} \middle|\, x_2, x_5 \in \mathbb{C} \right\}$$

5. Is the matrix B singular or nonsingular? Why? (15 points)

$$B = \begin{bmatrix} -1 & 2 & 0 & 3 \\ 1 & -3 & -2 & 4 \\ -2 & 0 & 4 & 3 \\ -3 & 1 & -2 & 3 \end{bmatrix}$$

Solution: Theorem NSRRI tells us we can answer this question by simply row-reducing the matrix. Doing this we obtain,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Since the reduced row-echelon form of the matrix is the 4×4 identity matrix I_4, we know that B is nonsingular.

6. Say as much as you can about the solution sets for the linear systems of equations described below. (15 points)

(a) 9 variables and 5 equations.

Solution: The system could be inconsistent. If it is consistent, then because it has more variables than equations Theorem CMVEI implies that there would be infinitely many solutions. So, of all the possibilities in Theorem PSSLS, only the case of a unique solution can be ruled out.

(b) Homogeneous, 6 variables, 8 equations.

Solution: By Theorem HSC, we know the system is consistent because the zero vector is always a solution of a homogeneous system. There is no more that we can say, since both a unique solution and infinitely many solutions are possibilities.

(c) Nonsingular coefficient matrix, not homogeneous.

Solution: Any system with a nonsingular coefficient matrix will have a unique solution by Theorem NSMUS. If the system is not homogeneous, the solution cannot be the zero vector (Exercise HSE.T10).

7. Consider the homogeneous system of linear equations $\mathcal{LS}(A,\,\mathbf{0})$, and suppose that $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$ is one solution to the system of equations. Prove that $\mathbf{v} = \begin{bmatrix} 4u_1 \\ 4u_2 \\ 4u_3 \\ \vdots \\ 4u_n \end{bmatrix}$ is also a solution to $\mathcal{LS}(A,\,\mathbf{0})$. (15 points)

Solution: Suppose that a single equationfrom this system (the i-th one) has the form,

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n = 0$$

Evaluate the left-hand side of this equation with the components of the proposed solution vector $\mathbf{v}$,

$$\begin{aligned}
&a_{i1}\,(4u_1) + a_{i2}\,(4u_2) + a_{i3}\,(4u_3) + \cdots + a_{in}\,(4u_n) \\
&\qquad = 4a_{i1}u_1 + 4a_{i2}u_2 + 4a_{i3}u_3 + \cdots + 4a_{in}u_n && \text{Commutativity} \\
&\qquad = 4\,(a_{i1}u_1 + a_{i2}u_2 + a_{i3}u_3 + \cdots + a_{in}u_n) && \text{Distributivity} \\
&\qquad = 4(0) && \mathbf{u}\text{ solution to }\mathcal{LS}(A,\,\mathbf{0}) \\
&\qquad = 0
\end{aligned}$$

So $\mathbf{v}$ makes each equation true, and so is a solution to the system.

Notice that this result is not true if we change $\mathcal{LS}(A,\,\mathbf{0})$ from a homogeneous system to a non-homogeneous system. Can you create an example of a (non-homogeneous) system with a solution $\mathbf{u}$ such that $\mathbf{v}$ is not a solution?