Math 232 Name: Key Dr. Beezer
Quiz VS Fall 2004

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points.

1. In the vector space of 2×2 matrices, M_{22}, determine if the set S below is linearly independent. (15 points)

$$S = \left\{ \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \right\}$$

Solution: Begin with a relation of linear dependence on the vectors in S and massage it according to the definitions of vector addition and scalar multiplication in M_{22},

$$\mathcal{O} = a_1 \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix} + a_3 \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2a_1 + 4a_3 & -a_1 + 4a_2 + 2a_3 \\ a_1 - a_2 + a_3 & 3a_1 + 2a_2 + 3a_3 \end{bmatrix}$$

By our definition of matrix equality (Definition ME) we arrive at a homogeneous system of linear equations,

$$\begin{aligned} 2a_1 + 4a_3 &= 0 \\ -a_1 + 4a_2 + 2a_3 &= 0 \\ a_1 - a_2 + a_3 &= 0 \\ 3a_1 + 2a_2 + 3a_3 &= 0 \end{aligned}$$

The coefficient matrix of this system row-reduces to the matrix,

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

and from this we conclude that the only solution is $a_1 = a_2 = a_3 = 0$. Since the relation of linear dependence (Definition RLD) is trivial, the set S is linearly independent (Definition LI).

2. Working within the vector space P_3 of polynomials of degree 3 or less, determine if $p(x) = x^3 + 6x + 4$ is in the subspace W below. (15 points)

$$W = \mathrm{Sp}\left(\left\{x^3 + x^2 + x,\ x^3 + 2x - 6,\ x^2 - 5\right\}\right)$$

Solution: The question is if p can be written as a linear combination of the vectors in W. To check this, we set p equal to a linear combination and massage with the definitions of vector addition and scalar multiplication that we get with P_3 (Example VS.VSP)

$$\begin{aligned} p(x) &= a_1(x^3 + x^2 + x) + a_2(x^3 + 2x - 6) + a_3(x^2 - 5) \\ x^3 + 6x + 4 &= (a_1 + a_2)x^3 + (a_1 + a_3)x^2 + (a_1 + 2a_2)x + (-6a_2 - 5a_3) \end{aligned}$$

Equating coefficients of equal powers of x, we get the system of equations,

$$\begin{aligned} a_1 + a_2 &= 1 \\ a_1 + a_3 &= 0 \\ a_1 + 2a_2 &= 6 \\ -6a_2 - 5a_3 &= 4 \end{aligned}$$

The augmented matrix of this system of equations row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

There is a leading 1 in the last column, so Theorem RCLS implies that the system is inconsistent. So there is no way for p to gain membership in W, so $p \notin W$.

3. M_{22} is the vector space of 2×2 matrices. Let S_{22} denote the set of all 2×2 symmetric matrices. That is

$$S_{22} = \left\{ A \in M_{22} \mid A^t = A \right\}$$

(30 points)
(a) Show that S is a subspace of M_{22}.

Solution: We will use the three criteria of Theorem TSS. The zero vector of M_{22} is the zero matrix, $\mathcal{O}$ (Definition ZM), which is a symmetric matrix. So S_{22} is not empty, since $\mathcal{O} \in S_{22}$.

Suppose that A and B are two matrices in S_{22}. Then we know that $A^t = A$ and $B^t = B$. We want to know if $A + B \in S_{22}$, so test $A + B$ for membership,

$$\begin{aligned} (A+B)^t &= A^t + B^t && \text{Theorem TASM} \\ &= A + B && A,\ B \in S_{22} \end{aligned}$$

So $A + B$ is symmetric and qualifies for membership in S_{22}.

Suppose that $A \in S_{22}$ and $\alpha \in \mathbb{C}$. Is $\alpha A \in S_{22}$? We know that $A^t = A$. Now check that,

$$\begin{aligned} \alpha A^t &= \alpha A^t && \text{Theorem TASM} \\ &= \alpha A && A \in S_{22} \end{aligned}$$

So αA is also symmetric and qualifies for membership in S_{22}.

With the three criteria of Theorem TSS fulfilled, we see that S_{22} is a subspace of M_{22}.

(b) Exhibit a basis for S and prove that it has the required properties.

Solution: An arbitrary matrix from S_{22} can be written as $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$. We can express this matrix as

$$\begin{aligned} \begin{bmatrix} a & b \\ b & d \end{bmatrix} &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \\ &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

this equation says that the set

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

spans S_{22}. Is it also linearly independent?

Write a relation of linear dependence on S,

$$\mathcal{O} = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$$

The equality of these two matrices (Definition ME) tells us that $a_1 = a_2 = a_3 = 0$, and the only relation of linear dependence on T is trivial. So T is linearly independent, and hence is a basis of S_{22}.

(c) What is the dimension of S?

Solution: The basis T found in part (b) has size 3. So by Definition D, $\dim(S_{22}) = 3$.

4. In $\mathbb{C}^3$, the vector space of column vectors of size 3, prove that the set Z (below) is a subspace. (15 points)

$$Z = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \; 4x_1 - x_2 + 5x_3 = 0 \right\}$$

Solution: The membership criteria for Z is a single linear equation, which comprises a homogeneous system of equations. As such, we can recognize Z as the solutions to this system, and therefore Z is a null space. Specifically, $Z = \mathcal{N}\left(\begin{bmatrix} 4 & -1 & 5 \end{bmatrix}\right)$. Every null space is a subspace by Theorem NSMS.

A less direct solution appeals to Theorem TSS.

First, we want to be certain Z is non-empty. The zero vector of $\mathbb{C}^3$, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, is a good candidate, since if it fails to be in Z, we will know that Z is *not* a vector space. Check that

$$4(0) - (0) + 5(0) = 0$$

so that $\mathbf{0} \in Z$.

Suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are vectors from Z. Then we know that these vectors cannot be totally arbitrary, they must have gained membership in Z by virtue of meeting the membership test. For example, we know that $\mathbf{x}$ must satisfy $4x_1 - x_2 + 5x_3 = 0$ while $\mathbf{y}$ must satisfy $4y_1 - y_2 + 5y_3 = 0$. Our second criteria asks the question, is $\mathbf{x} + \mathbf{y} \in Z$? Notice first that

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

and we can test this vector for membership in Z as follows,

$$\begin{aligned}
&4(x_1 + y_1) - 1(x_2 + y_2) + 4(x_3 + y_3) \\
&= 4x_1 + 4y_1 - x_2 - y_2 + 5x_3 + 5y_3 \\
&= (4x_1 - x_2 + 5x_3) + (4y_1 - y_2 + 5y_3) \\
&= 0 + 0 && \mathbf{x} \in Z,\ \mathbf{y} \in Z \\
&= 0
\end{aligned}$$

and by this computation we see that $\mathbf{x} + \mathbf{y} \in Z$.

If α is a scalar and $\mathbf{x} \in Z$, is it always true that $\alpha\mathbf{x} \in Z$? To check our third criteria, we examine

$$\alpha\mathbf{x} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}$$

and we can test this vector for membership in Z with

$$\begin{aligned}
4(\alpha x_1) - (\alpha x_2) + 5(\alpha x_3)& \\
&= \alpha(4x_1 - x_2 + 5x_3) \\
&= \alpha 0 && \mathbf{x} \in Z \\
&= 0
\end{aligned}$$

and we see that indeed $\alpha\mathbf{x} \in Z$. With the three conditions of Theorem TSS fulfilled, we can conclude that Z is a subspace of $\mathbb{C}^3$.

5. Suppose that V is a vector space. Prove that $-\mathbf{v} = (-1)\mathbf{v}$ for any $\mathbf{v} \in V$. (15 points)

   Solution: This is Theorem AISM and there is a careful proof there.