Show all of your work and explain your answers fully. There is a total of 90 possible points.

1. Consider the matrix $A$ below. (30 points)

\[ A = \begin{bmatrix}
1 & 2 & 0 & 2 & 0 \\
-1 & 1 & -3 & -2 & 3 \\
4 & 3 & 4 & 7 & -5
\end{bmatrix} \]

(a) Find a basis for the null space of $A$, $N(A)$.

(b) Find a basis for the range of $A$, $R(A)$.

(c) Find another basis for the range of $A$, $R(A)$, by using a substantially different method.
2. Let $S \subseteq \mathbb{R}^4$ be the set of vectors below. (12 points)

$$S = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 1 \\ 3 \end{bmatrix} \right\}$$

(a) Is $S$ an orthogonal set? An orthonormal set?

(b) Is $S$ a basis of $\mathbb{R}^4$?

3. Suppose $W$ is a subspace of $\mathbb{R}^5$ with dimension 3. Which of the following sets of vectors from $W$ are bases of $W$? (18 points)

(a) $\left\{ \begin{bmatrix} -9 \\ 8 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 20 \\ -13 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} -9 \\ 8 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 18 \\ -13 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ -5 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 7 \\ -5 \\ -3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 5 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -7 \\ 8 \\ 2 \\ 1 \\ -3 \end{bmatrix} \right\}$
4. Write a careful proof that \( W = \{(x_1, x_2, x_3) \mid 3x_1 - 5x_2 + x_3 = 0\} \) is a subspace of \( \mathbb{R}^3 \). (15 points)

5. Suppose that \( B = \{v_1, v_2, \ldots, v_m\} \) is a basis of the subspace \( V \subseteq \mathbb{R}^n \) (so in particular, \( m \leq n \)), and that \( w \in V \) is any vector from \( V \). Prove that there is exactly one set of scalars \( a_1, a_2, \ldots, a_m \) so that \( w \) can be written in the form \( w = a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_mv_m \). (15 points)